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**Pricing with markups under horizontal and
vertical competition**

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PRICING WITH MARKUPS UNDER HORIZONTAL AND VERTICAL COMPETITION

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ABSTRACT. We study the effect of competition on markups for producers of both substitutes and complements. We model a market for a single product, or product bundle, where producers compete to provide all or some portion of the product. In our game, each firm adopts a price function proportional to its per-unit costs by deciding on the size of a markup. Customers then choose a set of providers that offers the lowest combined price. For quadratic production costs and general series-parallel market structures, a producer's optimal markup corresponds to the price in a redefined market, which we derive explicitly. We characterize equilibrium markups for inelastic and elastic demand. When bundle demand is inelastic, a unique equilibrium exists if and only if the market has a 3-edge-connected network structure. We study comparative statics of equilibria with respect to changes in the market design. When production is divided among decentralized firms that compete vertically, markups increase throughout the market. Reduced competition in the market for any bundle component leads to a higher bundle price, but social efficiency in component markets need not coincide with overall efficiency of production. Furthermore, the effect on an individual producer's profit is seen to depend on its market position.

KEYWORDS. Noncooperative games, Oligopoly, Bundling, Supply function equilibrium.

1. INTRODUCTION

Classical models of competition, through either prices or production quantities, have focused predominantly on markets of a single good. In this setting, producers of substitutes, either perfect or imperfect, compete *horizontally* for the same pool of customers. Recently, there has been increasing recognition that in some industries competition among customers has a significant combinatorial component, beyond the scope of traditional single-market models. In such industries, producers whose goods may be purchased in combination compete *vertically*. Vertical competitors, while sharing the same customer base, compete for the ability to extract larger markups, and thus claim a larger stake of profits from their combined product. In this paper, we study these markups through a framework where each producer chooses a price schedule, having in mind both the actions of vertical and horizontal competitors. Each producer sets its pricing schedule according to a price function that is specified as a constant percentage markup over per-unit production costs. Producers set the pricing schedule for individual products and customers choose a bundle of products at minimum price. Customers are interested in bundles composed of products given by any path of a series-parallel network. Producers on parallel links compete horizontally, while connections in series represent vertical relationships.

Industries with our structure include those where a physical or geographic network is explicitly present, as in the airline industry, as well as those where a network structure is defined implicitly by the available bundles. Our model allows for many infinitesimal customers, or a single, centralized

buyer. This latter case applies to decentralized assembly supply chains, where a monopsonistic manufacturer purchases components from multiple suppliers. The assembly paradigm applies naturally to the increasingly modular production of, e.g., automobiles and airplanes, among other products. Granot and Yin (2008) discuss further applications related to the bundling of media products, transportation, and health care services.

In all of these examples, there is a question of the extent to which producers of each product in the bundle may exercise market power. For example, according to Dedrick et al. (2010), who study supplier data for the Hewlett-Packard nc6230 Notebook PC, Toshiba's gross profit margin for its display component was 28% of net sales, while Intel's processors had a gross profit rate of 59%, indicating a substantially stronger market position. Intel's position reflects the strength of its proprietary technology and brand status, which make it difficult to replace in the Notebook PC supply chain. Additionally, each firm's profit margins depend on the structure of the complementary markets. For example, as AMD emerges to challenge Intel in the CPU market, prices may shift throughout the industry (see Casadesus-Masanell et al. (2010) for a discussion regarding Microsoft Windows). To the extent that intense competition will lower the price of CPUs, Toshiba stands to benefit with increased sales. What, however, is the effect of a stronger AMD on Toshiba's market power?

For a general set of market configurations, allowing varying degrees of integration among competitors, our formulation addresses these questions of market power. Furthermore, our analysis of the optimal producer markups provides a useful decomposition of the inter-component competitive effects illustrated above. For a given producer and its complementary market, there is a vertical effect that shifts the residual demand curve up as competition is added (Toshiba benefits from reduced CPU prices), and an additional horizontal effect, that increases its elasticity, and so lessens equilibrium profit margins (a stronger AMD *weakens* Toshiba). Our decomposition represents these two effects explicitly for a producer facing an arbitrary configuration of vertical and horizontal competition, as captured through a series-parallel network representation. We then exploit these structural properties to compute equilibrium markups, and compare the size of both competitive effects for various changes to the market structure.

In contrast to models of purely price competition, here market power derives from each producer's knowledge that competitor prices will adjust along a price schedule in response to its own pricing decision. When competitor markups are high, the response is more dramatic, and outcomes more closely reflect quantity competition. Notice that strict quantity competition does not provide an obvious mechanism for dividing revenues among the products in a bundle. By considering price functions, we capture the producers' reactions to varying demand levels, which in turn determines the distribution of revenues when the market clears. In practice, the reactions that we model are often embodied implicitly in a producer's pricing strategy. While in some cases, such as in electricity or bond markets, there is a market mechanism that allows an explicit price function to be posted, we are in general interested in any case where producers react to competitors in part through price adjustments. Klemperer and Meyer (1989) provide further discussion of the organizational mechanisms by which such price functions may be implicitly defined.

This work highlights the strategic interaction between producers whose output may be bundled together. There are several threads of literature involving complementarities. In the model of Cournot (1838), price-setting monopolists of copper and zinc, which are combined to make brass, split profits equally among them. Multi-market oligopoly theory, dating back to Bulow et al. (1985), also incorporates complementary goods, but not within the bundling framework we consider. Closer to our setup is the literature on decentralized assembly systems (see, e.g., Granot and Yin (2008), Wang (2006), and the references therein), in which an assembler purchases a set of components from multiple strategic suppliers. In particular, Jiang and Wang (2010) model competition within individual component markets, which relaxes Cournot's equal-profit result when competition is

asymmetric. In their model, competition is Bertrand, as suppliers compete by fixing their wholesale price. Constant marginal costs ensure that a single firm produces each component. Others, such as Nagarajan and Bassok (2008), look at alliance-formation among suppliers. We consider a fixed network structure, but the generality of our network should be useful for evaluating such alliances. In an airline context, Lederer (1993) studies competitive network design, and shows that equilibrium prices may not exist if legs from competing airlines can be bundled. Netessine and Shumsky (2005) look at management of seating inventory on vertically competing flight legs. Finally, the model of Casadesus-Masanell et al. (2010) expands on Cournot’s original model by allowing competition and vertical differentiation in one of the markets.

There is also a growing literature on competition in networks, motivated largely by applications involving a physical network. The focus has been the role of prices in guiding users towards efficient paths through the network. There is a large literature concerning centralized pricing, and a more recent body of work on price competition by decentralized firms. See, e.g., Acemoglu and Ozdaglar (2007a), Correa et al. (2008), Acemoglu et al. (2009), Johari et al. (2010), Johari and Tsitsiklis (2010) for the case of substitutes, or Acemoglu and Ozdaglar (2007b), Chawla and Roughgarden (2008), Papadimitriou and Valiant (2010) for more general market structures. A feature of physical networks is often that customers experience costs due to congestion, with the effects increasing in the number of customers sharing a path. Our model is more in line with traditional models of competition in that customers experience no costs outside of the price that is paid to the producer. However, producers themselves experience costs that are marginally increasing, and pass this cost structure on to the customers through their price schedules. In this way, demand is encouraged to spread across multiple paths as in a network with congestion.

Competition in price functions is an example of supply function equilibrium (Grossman 1981, Klemperer and Meyer 1989). The supply function equilibrium concept is a generalization of Cournot and Bertrand models of competition. In each of these cases, producers commit to either prices or production quantities before observing their competitors’ choices, leaving only one of these as a possible lever for responding to the market. In the case of supply function equilibria, producers choose a function relating the price to the quantity produced (i.e., the inverse of the price function). Then, after all such functions have been chosen, the firms can adjust to the market conditions by choosing a point along this supply function. In equilibrium, each producer sells according to a single quantity/price combination, selected from its supply function so that the market clears. In this way, supply function equilibria model the common scenario where both price and quantity are adjusted in response to the market conditions. Klemperer and Meyer (1989) show that, in the case of a duopoly, competition in supply functions leads to equilibrium prices and quantities intermediate between those of Bertrand and Cournot competition. In related research, Akgün (2004) analyzes mergers of firms with quadratic cost functions in a market of substitutes using the framework of Klemperer and Meyer (1989). For a market of substitutes with a deterministic demand and nonlinear cost functions, Correa et al. (2008) identify an equilibrium where price functions have the same structure as cost functions and study its properties, particularly the efficiency of the resulting market. In this work, we focus on more general market structures and use supply functions to smooth price/quantity competition in a tractable way for markets involving both horizontal and vertical competition.

As in Correa et al. (2008), our producers are assumed to select a price function that maintains the same shape as their cost function. This model is consistent with cost-plus pricing policies that are often employed in practice. Furthermore, such behavior simplifies analysis by reducing each producer’s decision to a single parameter, which we call its markup. We refer to supply function equilibria where each firm is restricted to playing a markup over its cost function as *markup equilibria*. As additional motivation for the restriction to markup equilibria, we show that even when producers may choose any non-decreasing price function, any markup equilibrium

remains an equilibrium in the unrestricted game. This generalizes a result of Correa et al. (2008) for the case of substitutes. In addition, Klemperer and Meyer (1989) consider the game with general supply functions in the symmetric duopoly case, and show that of the many equilibria existing when demand is deterministic, uncertainty eliminates all but the unique markup equilibrium. See Appendix A for our result on the robustness of markup equilibria.

An important feature of our model is that per-unit costs of production, and thus price functions, are increasing, so that demand is allocated to multiple competitors where they exist. Marginally-increasing costs are a common assumption in industries where capacity is constrained or costly to increase in the short term. Even when long-term capacity investments exhibit economies of scale, it is often the case that quantity decisions for specific product lines are made subject to prevailing capabilities. Short-term adjustments, such as bringing on temporary workers, will then entail additional marginal expense. In practice, the number of competitors producing positive quantities is often limited by the presence of substantial fixed costs of production. In our model, there are no fixed costs. Rather, we consider that in the time scales we study, long-term decisions of market entry and capacity investment are made exogenously. We thus include only those competitors that actually produce, and consider pricing decisions subject to capacity restrictions. This assumption allows a tractable characterization of the demand allocation through a system of equations, rather than the complementarity problem that results when entry decisions are incorporated. Other literature assuming negligible fixed costs includes, e.g., Acemoglu and Ozdaglar (2007a) and Johari and Tsitsiklis (2010).

The equilibrium outcome is simplified further by an assumption that all producers face a quadratic cost function. While the shape is thus constant, we allow for heterogeneity among producers by applying an efficiency parameter that scales each cost function according to the specifics of the firm’s production technology. The restriction to quadratic cost functions is used to enable a closed-form derivation of a producer’s optimal markup, which sharpens our insights relating market structure and producer markups. In particular, the link between optimal markups and the degree of horizontal competition is made explicit. Much of our analysis does in fact extend to general monomial cost functions, and this case is discussed further in the conclusion. Linear marginal costs, as we impose in the body, are often assumed for tractability in the literature on supply function equilibria. Baldick et al. (2004) categorize work in the area into those assuming duopoly (e.g., Green and Newbery (1992), Laussel (1992)) and those assuming linear marginal costs (e.g., Green (1996), and more recently Baldick et al. (2004), Akgün (2004)). Recent exceptions are Correa et al. (2008), who consider convex cost functions initially before restricting to a monomial form, and Johari and Tsitsiklis (2010), who allow convex costs but place other restrictions on the form of supply function chosen. In a broader context, multi-stage games often require relatively restrictive assumptions regarding costs to ensure tractability (see e.g., Engel et al. (2004), Acemoglu and Ozdaglar (2007a), Wichensin et al. (2007), Xiao et al. (2007), Johari et al. (2010)).

In this paper, we present the first study of supply function equilibria in markets with both substitutes and complements, analyzing the two-stage game in which producers select price functions, anticipating the allocation game occurring in the second stage. We observe that markups of vertical competitors are strategic complements, and that a sufficient degree of horizontal competition is needed for markups to stabilize to an equilibrium. Formalizing this, we present a necessary and sufficient condition for the existence of equilibria, and show that the equilibrium is unique when it exists. For a fixed, inelastic demand, an equilibrium exists only in networks that are 3-edge-connected (see e.g., Ahuja et al. (1993) for background on graph connectivity). Notice that this condition depends entirely on the topology of the network, and is independent of the cost parameters. For a network of substitutes, this is equivalent to requiring at least 3 producers to compete in the market (Correa et al. 2008). Interestingly, this matches results of existence of equilibria in related models (Kyle 1989, Johari and Tsitsiklis 2010). For general series-parallel networks, this

condition rules out the case in which two producers within a bundle act as ‘monopolies’ in that no other firm can replace them in that bundle. A similar problem was discussed in the network competition model of Acemoglu and Ozdaglar (2007b), and both scenarios are reminiscent of double marginalization, which is widely recognized as a source of inefficiency. When demand is elastic, the outside option provides sufficient horizontal competition, but a weaker existence condition is still needed to address the potential for vertical instability.

The best-response functions of producers have a highly intuitive structure: the per-unit price equals the per-unit cost plus a markup whose functional expression depends only on the markups of everybody else. For each producer, we provide a procedure by which the network structure is pivoted to represent its unique set of substitute bundles. Equilibrium prices in this redefined market then measure the producer’s market power and determine its markup. For a fixed set of competitor markups, the best-response markup decreases with the introduction of horizontal competitors and increases with the number vertical competitors required to produce a bundle. Both relationships are intuitive, and serve to validate our representation of market power. In particular, the effect of additional complementarity is consistent with that of Wang (2006) for assembly systems.

The generality of our network structure sheds light on some intriguing inter-market relationships. Our sensitivity analysis of equilibria first shows that an increase in any producer’s costs will increase the markups of all producers in the network. While this is intuitive for horizontal competitors, it may contrast with intuition about the merits of enabling complementors to innovate. Furthermore, while it is often thought that intense competition among its complementors will yield a producer the greatest market power, the result indicates that this may not be the case. We do note that, as our decomposition approach makes clear, reduced pricing power can be balanced by increased market share, yielding a positive net effect in these cases. These simultaneous effects make analysis of producer profits more complex than that of markups. As in Jiang and Wang (2010), the addition of competition within component markets allows an unequal division of profits in our model. However, in markets with price flexibility and multiple active producers, we observe some additional dynamics. While in their model, each producer benefits from an efficiency gain that increases downstream competition, we see that such changes can alter the balance of power between direct competitors. In particular, we show that while the most efficient producers always gain share in this case, their less efficient competitors may lose share and so see their profits decrease. Interestingly, this suggests that while market leaders are likely to encourage any innovations by complementors, secondary players may actually benefit by preserving inefficiency elsewhere in the supply chain.

We explore the effect of network structure on overall market efficiency. Intuitively, we show that the bundle price increases when direct competitors merge, but mergers of vertical competitors actually decrease the bundle price. In addition, for production that is split among vertical competitors, we find that if one component market is dominated by a monopoly, it is most inefficient when the cost parameter for that component is small. For an assembler that can structure its network of suppliers, this suggests that a strategy where components are relatively equal in value may be favorable to an asymmetric split. Furthermore, when production costs for a component do not increase quickly, it is beneficial to cultivate multiple suppliers for the purpose of distributing market power. Besides bundle price, we also compare the production cost realized at equilibrium to the cost when customers are allocated optimally with respect to producer cost functions. Here, the loss of efficiency results from the fact that markups distort the cost structure, leading customers to purchase less from those producers with high markups than they would in an optimum allocation. In contrast to what we observe with price, we discuss an example where, surprisingly, a horizontal merger in the market for a component can actually reduce the overall cost of producing a bundle.

We proceed in Section 2 with a description of our model and analysis of the second-stage game that follows the choice of markups. Section 3 then discusses equilibria of the markup game played

by producers. Section 4 looks at sensitivity of equilibrium outcomes for individual producers, while Section 5 studies market efficiency with respect to mergers. In Section 6, we compare our full model to a model that considers only direct competitors, and discuss the benefit to a component producer of understanding its vertical competition. We then close in Section 7 with a look at some future directions.

2. MODEL AND GAME STRUCTURE

We model a market for complementary goods by considering demand for a single good that we will call a bundle. Customers face multiple options for purchasing a bundle, and while each is equivalent in the eyes of the customer (they are considered perfect substitutes), they may be the result of production from a number of separate producers, each selling some portion of the good. We recognize that there need not be any single definitive way to divvy up production of a bundle, and so our model is general enough to allow each purchase option to be subdivided among any number of producers. Furthermore, each subdivision defines a production niche that multiple producers may compete to fill. Finally, among the various options for filling any particular niche, we consider that each may be further subdivided in some fashion and split among more specialized producers.

In general, we look at markets that take a series-parallel (SP) structure. The class of SP networks are exactly those that can be constructed recursively through link subdivisions in series and in parallel. That is, through repeated application of the operations $\text{DivS}(\cdot)$ and $\text{DivP}(\cdot)$, pictured in Figure 1. $\text{DivS}(a)$ subdivides a link a into two links, connected in series, with a newly created node joining the head of a' and tail of a'' . $\text{DivP}(a)$ subdivides a link a into two links, a' and a'' , each connecting the same two nodes as the original link a . Seeing as we do not limit the number or arrangement of subdivisions, SP networks provide a great deal of generality.

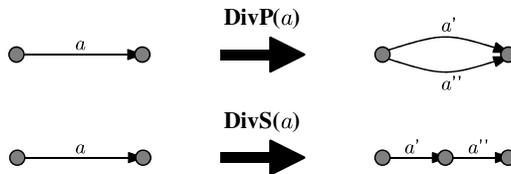


FIGURE 1. Operations defining a series-parallel network.

We model the set of available purchase combinations as paths from the source s to the sink t of an SP network, G , comprising a set of links $A_G = \{1, \dots, n\}$. Each link $a \in A_G$ represents a producer, and each path through G a bundle that customers may purchase. Thus, denoting the set of available bundles by $\mathcal{B} := \{B_1 \dots B_m\}$, we say for producer $a \in A_G$, that $a \in B_i$ if link a appears along path B_i in the network representation. In this way, the network defines a mapping of producers to purchase bundles. Each customer chooses a complete bundle, so that $\sum_{i=1}^m f_i = 1$ where f_i is the proportion of customers choosing bundle B_i . Then, the proportion of demand produced by producer a , is equal to $x_a = \sum_{B_i \ni a} f_i$. Because we interpret f_i and x_a as proportions, the total demand is normalized to one. We assume that individual customers are small so that demand is divisible among them, and each acts as a price taker. Although the discussion is for the case of inelastic demand, most results extend to the elastic case (see Section 3.2).

Figure 2 illustrates this network formulation for a stylized model of competition in the market for computers. The market contains five producers, and customers may choose a CPU-monitor combination from separate producers or opt to purchase an integrated model containing both. There is a single option for purchase of an integrated computer, but customers can create any of

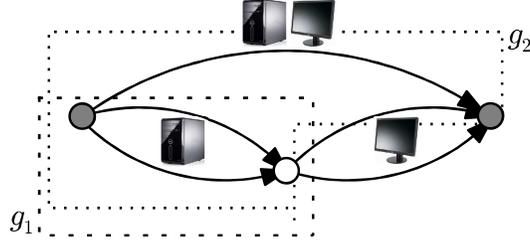


FIGURE 2. Example of SP network structure applied to the market for computers. The market g_1 for CPUs is a submarket, as defined in Section 2.1. The network g_2 is not a submarket.

four distinct bundles by choosing from among duopolists in the markets for CPUs and monitors, respectively. In this case, in addition to the usual horizontal competition, manufacturers of CPUs and monitors compete with each other vertically so as to determine the profit each gets from the bundle they jointly provide¹. Thus, manufacturers of CPUs and monitors face conflicting interests. While they both benefit from the demand induced by low prices on their joint offering, each seeks to maximize their own share of the profit. Note that the graphical ordering of component markets is arbitrary, and not indicative of any temporal ordering. In fact, the game we consider is static.

The per-unit production cost for each producer $a \in A_G$ is a function $u_a : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ that depends on the production level x_a . We assume that all producers make use of similar ‘technology’ but some are more efficient than others. This is modeled by a cost function of the form $u_a(x_a) := c_a u(x_a)$ where the function $u(x_a)$ is an indication of the industry’s unit cost for production level x_a , and the parameter $c_a > 0$ measures the efficiency of producer $a \in A_G$. We assume that per-unit costs are linear, i.e., $u(x) = x$ (see Section 2.3). More generally, the model may assume that u is increasing, differentiable, and bijective (i.e., evaluates to zero at zero and grows to infinity). Furthermore, $xu(x)$ is convex; in other words, industries face increasing marginal production costs, which is the case, e.g., when labor or production capacity is scarce or when there is congestion. Putting all the elements together, the total cost to producer a of producing x_a units is $\kappa_a(x_a) := x_a u_a(x_a) = c_a x_a u(x_a)$, which is quadratic in this paper, and convex in general.

We consider a two-stage game, where producers determine a pricing structure in the first stage, and customers choose a bundle of producers to purchase from in the second stage. In the first stage, producers commit to a price function $p_a(x_a)$ specifying the per-unit price to be charged at a specific level of production. Thus, both prices and production quantities are determined in the second stage, where the market clears. We assume that the price of a bundle is additive so that a customer purchasing bundle B_i pays a total of $\sum_{a \in B_i} p_a(x_a)$. Note that in the case of complementary items produced by the same producer, we would model purchase of both items by a single link. Thus, we are assuming additive pricing here only in the case of items purchased from competing producers. A critical feature of this structure is that the price a customer pays for a unit of production from producer a depends on the total quantity that producer a produces, which itself is dependent on the consumption choices of all customers. This gives the second stage its interpretation as a game between customers.

We simplify the first-stage game by restricting the set of price functions a firm may choose. We consider only *markups*, in the sense that the producer a ’s price function $p_a(x_a) = \alpha_a u_a(x_a)$ for some positive factor $\alpha_a \geq 1$ that is chosen by the producer. We interpret α_a as a markup, due

¹To allay confusion, we note the orientation of our diagrams, in which demand flows left to right, is such that producers who compete ‘vertically’, by virtue of the complementary nature of their products, are aligned horizontally on the page, and vice versa.

to the fact that α_a represents the ratio of price to production cost for producer a^2 . Within this framework, the shape of all price functions is determined exogenously through the cost structure, and producers compete by selecting a single parameter. This is not as restrictive an assumption as it may seem. Even in the setting where producers may choose any non-decreasing price-function, it can be shown that while there are in general many equilibria for the game, at any equilibrium in price-functions it is a best response to play a price function that is a markup of the producer's cost function. This robustness result was shown for a network of parallel links in Correa et al. (2008). In Appendix A, we present an extension to the setting of SP networks. By exogenously setting the shape of all price functions as we have, we allow each to be described completely by a single *price multiplier* $w_a := c_a \alpha_a$. The actual unit price for product a is then given by $w_a u(x_a)$.

We seek to analyze the assignment of demand to specific producers. An assignment is described through either consumption decisions, using the vector $\vec{f} \in \mathbb{R}_+^m$, or through production quantities, as represented by the vector $\vec{x} \in \mathbb{R}_+^n$. The heterogeneity in our problem is across producers only, and so we will be primarily concerned with the production assignment \vec{x} . Note that for a given assignment \vec{x} , there may be multiple consumption allocations that give rise to \vec{x} . In particular, when we discuss uniqueness of an optimal or equilibrium production assignment, this need not imply uniqueness of the consumption assignment. We denote the set of possible production-consumption pairs by

$$\mathcal{F} := \left\{ (\vec{x}, \vec{f}) \in \mathbb{R}_+^{(n+m)} : \sum_{i=1}^m f_i = 1, x_a = \sum_{B_i \ni a} f_i \forall a \in A_G \right\}.$$

We say that a production allocation \vec{x} is feasible if there exists a consumption assignment \vec{f} such that $(\vec{x}, \vec{f}) \in \mathcal{F}$.

2.1. Submarket Structure. We now define the concept of a submarket. It will be helpful to introduce the composition operations $S(\cdot)$ and $P(\cdot)$, each of which takes as input a set \mathcal{G} of SP networks, and returns a single SP network. In the case of $S(\mathcal{G})$, the input networks are composed in series with the sink of one network doubling as the source node of the next. In the case of $P(\mathcal{G})$, the input networks are composed in parallel so that all share a common source and sink.

As our market connects the source and sink nodes of an SP network, so a *submarket* is defined by a subnetwork connecting two nodes of G . Formally, a submarket g , composed of producers A_g , is a connected subnetwork of G , with two terminal nodes, a source and sink, chosen from among the nodes in G , and the property that for any non-terminal node in g , all incident links are included in g as well. See Figure 2 for an example. The submarket g is self-contained in that it defines a product offering such that the output of any producer within g can be purchased only as part of that larger offering. The full market G is a submarket, as is any individual producer a . The flow into the source node of g represents the demand for the product this submarket produces. Were this quantity fixed, then competition on g would fit the form of our general model. As it is, a submarket strictly smaller than G faces an elastic demand, decreasing in the price of its offering. The price-sensitivity of this demand is determined by the price functions chosen by producers in $A_G \setminus A_g$.

The SP structure of G dictates that submarkets are arranged in a nested fashion. Each submarket g can be characterized as either a *series submarket*, indicating that $g = S(\mathcal{G})$ for some set \mathcal{G} of submarkets, or a *parallel submarket*, composed as $g = P(\mathcal{G})$. We can give the set \mathcal{G} of component markets comprising g an explicit name, denoting this set by $\psi(g)$. To avoid ambiguity, we require when g is a series submarket that all elements of $\psi(g)$ be parallel submarkets, and vice versa, so that $\psi(g)$ represents the largest (by cardinality) set of submarkets from which g can be formed in

²One could also consider additive markups to the cost. This would give rise to models related to the area of congestion pricing. For background on this literature, see e.g., Lawphongpanich et al. (2006).

a single composition. In defining $\psi(\cdot)$, we have implicitly defined a tree structure that captures all submarkets; see Figure 3 for an example. Beginning with G as the root, $\psi(\cdot)$ determines a set of successors for each node. Every submarket appears as a node in this tree representation, with each producer $a \in A_G$ appearing as a leaf node. By convention, we will think of individual producers as series (parallel) submarkets, when their predecessor is parallel (series). We can assume, without loss of generality, that G is a parallel submarket, because markets in series, subject to a fixed demand, have no interaction and can thus be considered independently.

We call this tree structure the *submarket representation* of G . It is clear from the submarket representation that any properties possessed by a single-producer submarket, and which are preserved by the operations $S(\cdot)$ and $P(\cdot)$, can be attributed to G . This type of induction argument will be helpful in the remainder. Secondly, the submarket representation provides a means of formalizing the position of a submarket. Each submarket is uniquely represented as the endpoint of a nested sequence of submarkets, beginning with G . For submarkets g, g' with $A_g \subseteq A_{g'}$, let $\psi_g(g')$ be a restricted form of the mapping $\psi(\cdot)$, returning only the component market of g' that contains g . Then $\nu(g) := (G, \psi_g(G), \psi_g^2(G), \dots, \psi_g^{d_g}(G))$ returns the path through the submarket tree to g . Here d_g is the depth of g in the submarket tree, and $\psi_g^{d_g}(G)$ is equal to g itself. The elements of $\nu(g)$ are increasingly specific descriptions of the submarket's relative position. For instance, using the model of Figure 2, if producer a is a CPU manufacturer, then $\psi_a(G)$ returns the market for CPU-monitor combinations, and $\psi_a^2(G)$ is the market for CPUs, labeled g_1 . The full submarket representation is illustrated in Figure 3.

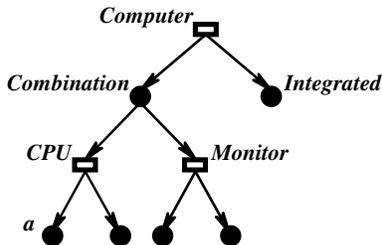


FIGURE 3. The submarket representation of the computer market in Figure 2.

For an arbitrary vector $\vec{v} \in \mathbb{R}^n$ defined on the full set of producers, we use the notation \vec{v}_A for the vector restricted to some set $A \subseteq A_G$. When g is an SP network representing some market, we abuse notation by referring directly to \vec{v}_g , with the understanding that this vector contains values for producers in A_g . In this respect, for two markets g and g' , we have $g' \subseteq g$ if $A_{g'} \subseteq A_g$, and $g \setminus g'$ denotes the set of producers contained in A_g , but not in $A_{g'}$.

2.2. Optimal and Equilibrium Assignments. To quantify the quality of an assignment, we consider the total production cost $C(\vec{x}) := \sum_{a \in A} c_a x_a u(x_a)$ as a social cost function. This function captures whether customers are matched to the producers that are most efficient. Notice that payments are not considered in this function because they are internal transfers. The socially optimal assignment, $\vec{x} = x^{\text{OPT}}$, is the unique production assignment minimizing $C(\vec{x})$. In other words,

$$(x^{\text{OPT}}, f^{\text{OPT}}) := \arg \min_{(\vec{x}, \vec{f})} \left\{ C(\vec{x}) : (\vec{x}, \vec{f}) \in \mathcal{F} \right\}. \quad (1)$$

The production assignment is unique because $u(\cdot)$ is such that $x_a u(x_a)$ is strictly convex for all $a \in A_G$.

An equilibrium for producers is a vector of markups $\vec{\alpha}$ that maximizes the profits of all producers simultaneously, and an equilibrium for customers is an assignment \vec{f} such that all customers are buying at minimal price. These two games are played sequentially, making it a Stackelberg game.

It will be convenient to think of the producers as setting price multipliers, leaving markups defined implicitly. So, in the markup game, producers first choose \vec{w} , followed by a second stage in which customers determine $\vec{f}(\vec{w})$, and consequently determine a production assignment, denoted $\vec{x}(\vec{w})$. For a fixed \vec{w} , producer a realizes profits

$$\pi_a(\vec{w}) := (w_a - c_a)x_a(\vec{w})u(x_a(\vec{w})). \quad (2)$$

The equilibrium conditions imply that a tuple (\vec{w}, x^{NE}) representing the two stages is at equilibrium if and only if $x^{\text{NE}} = \vec{x}(\vec{w})$, and

$$w_a \in \Phi_a(\vec{w}_{-a}) := \arg \max_{w \geq 0} \{\pi_a(w, \vec{w}_{-a})\} \text{ for all } a \in A_G \quad (3)$$

where $\Phi_a(\cdot)$ is the best response function of producer a to the price multipliers of all other producers, denoted by \vec{w}_{-a} . Here, the second stage assignment $(\vec{x}(\vec{w}), \vec{f}(\vec{w})) \in \mathcal{F}$ is defined for an arbitrary vector \vec{w} and satisfies the condition

$$\sum_{a \in B_i} w_a u(x_a(\vec{w})) \leq \sum_{b \in B_j} w_b u(x_b(\vec{w})) \quad (4)$$

for all $B_i, B_j \in \mathcal{B}$ such that $f_i(\vec{w}) > 0$. The above inequality says that in any equilibrium of the second-stage game, all bundles sell, if at all, at a single minimal price.

The uniqueness of equilibrium markups is established later on, but at this point it is clear that $\vec{x}(\vec{w})$ is unique for any \vec{w} because the function $u(\cdot)$ is strictly increasing (Beckmann et al. 1956). Notice that price distortions driven by producers with market power, as well as potential negative externalities in the second stage, make it such that the markups \vec{w} may not give rise to the most efficient equilibrium assignment. Rather, it is likely that $C(x^{\text{NE}}) > C(x^{\text{OPT}})$.

2.3. Linear Unit Cost Functions. Recall our assumption that total cost functions are quadratic. In this case, we have that $u(x) = x$. Hence the total cost for producer a has the form $\kappa_a(x_a) = c_a x_a^2$. From a technical point of view, this assumption allows us to explicitly characterize the optimal assignment and the unique assignment corresponding to a given vector of markups. Indeed, in this situation $C(\vec{x})$ is a convex function and the absence of any fixed costs ensures that all producers are active under both assignments. An immediate consequence is that the inequalities in the conditions of (4) become tight, and so all bundles sell for the same price. Furthermore, the restriction to linear unit costs is sufficient to ensure that customers are efficient in the sense that for fixed \vec{w} , $\vec{x}(\vec{w})$ minimizes $\sum_{a \in A_G} x_a w_a u(x_a)$. Thus, their behavior in the game is consistent with that of a centralized buyer with the ability to split consumption optimally among the bundles.

Seeing as customers behave efficiently, any inefficiency in the assignment is the result of distortion of the true cost functions due to producer markups. It follows that when markups are not distortionary, the equilibrium assignment will match the socially optimal assignment. (Proofs of all statements are provided in the appendix.)

Proposition 1. *The unique socially optimal assignment x^{OPT} is equal to $\vec{x}(\vec{c})$.*

Thus, the optimal assignment matches the second stage equilibrium that results when producers charge their actual costs without markup. In the next section, we characterize the second-stage equilibrium assignment for any fixed set of price functions, which will include the optimal assignment as a special case.

2.4. Analysis of Second Stage. In this section we present a precise functional form of the second-stage assignment $\vec{x}(\vec{w})$, where \vec{w} is a fixed vector of price multipliers. Its own markup aside, each firm's production increases with the markups of substitute products, while abating in response to those of vertical competitors. The parametrization below encapsulates both of these effects, resulting in an efficient characterization of the production assignment.

To start, we introduce the *network price multiplier* $R_g(\vec{w}_g)$, which generalizes w_a to a submarket g . When a demand of x_g is assigned to g according to (4), the market price for a bundle is $R_g(\vec{w}_g)x_g$. As \vec{w} is fixed prior to the assignment game, we use the notation R_g with the understanding that the multiplier reflects the combined effects of a set of price functions selected by individual producers in the first-stage game. Since demand for market G is normalized to one unit, R_G is also the equilibrium price of a bundle under \vec{w} .

For an individual producer, R_a is equal to w_a . For a larger submarket g , R_g depends ultimately on the proportions in which customers choose from among the bundles in g . A full characterization is obtained inductively by:³

$$R_{S(g)} = \sum_{g \in \mathcal{G}} R_g, \quad \text{and} \quad R_{P(g)} = (\sum_{g \in \mathcal{G}} 1/R_g)^{-1}, \quad (5)$$

which follows from (4) since customers allocate to parallel submarkets in inverse proportion to their price multipliers.

The nested structure of our model allows us to characterize inductively the total demand of a given producer. Let $\nu_P(a) = (G_1, G_2 \dots G_d)$ be the sequence of parallel submarkets within which producer a is nested (these are the odd elements of $\nu(a)$, beginning with G). Then, the proportion of customers that choose a submarket $\psi_a(g)$ of g , out of those customers that choose g , is given by the fraction $R_g/R_{\psi_a(g)}$. Multiplying, we get that the demand for producer a is $x_a(\vec{w}) = \prod_{g \in \nu_P(a)} R_g/R_{\psi_a(g)}$. We say that a producer a *spans* the market if link a connects s and t directly. In this case, all other bundles are substitutes for a , and x_a is increasing in the multipliers of all competitors. If, on the other hand, producer a faces vertical competition, the residual demand for product a is shifted downwards as the markups on complementary items increase. Both effects can occur for a competitor $b \neq a$, and so the impact of w_b on $x_a(\vec{w})$ is not clear *a priori*.

2.5. Producer Best Response Function. The above expression of $x_a(\vec{w})$ encodes producers' demand, but is hard to manipulate directly to understand how producers will set their markups. In this section, we provide an alternative formula that is more amenable to analysis. In particular, we now express $x_a(\vec{w})$ in terms of aggregate measures of the horizontal and vertical competition faced by a . Our approach is to redefine the market by pivoting G so that the nodes incident to a become the source and sink. In this reformulation, denoted $G \odot a$, a spans the market and all competition with a is horizontal. To interpret, the market spanned by a is one in which all customers come to market in possession of a bundle that is perfectly complementary to a . To complete their bundle, customers may purchase from a or one of its direct competitors. In addition, there is the option to 'sell back' the complementary items and purchase a new bundle. Any such action can be represented as a path through $G \odot a$. In the course of pivoting G , any complementary links to a ; i.e., those on a path from s to a or a to t , are reversed in direction to reflect that these products are sold back to producers at the prevailing market price. Any combination of sales/purchases that forms a path through the pivoted network will leave the customer with a complete bundle, and is in effect a perfect substitute to a . Accordingly, we call the network created by removing a from $G \odot a$, the *substitute network* for producer a . The substitute network is denoted by $G \ominus a$, and its construction is demonstrated in Figure 4. The example in (c) contains vertical competition and so requires pivoting.

The uniqueness of the niche that producer a fills will determine the multitude of paths in $G \ominus a$, and play a key role in determining market power. A measure of this is $R_{\ominus a}$, defined for $G \ominus a$ according to (5). If producer a is a monopolist, then $G \ominus a$ is empty, indicating that customers have

³Equation (5) matches that used for electrical circuits to compute the equivalent resistance when placing resistors in series and parallel. Ohm's law, *Voltage = Current · Resistance*, is analogous to the price function $p_a = x_a R_a$. Although the equations describing both systems are identical, the difference is that we impose a nonnegativity restriction on flows, whereas in electricity networks this is not needed. It is precisely those restrictions that complicate the analysis of a general network as we will discuss in Section 3.5.

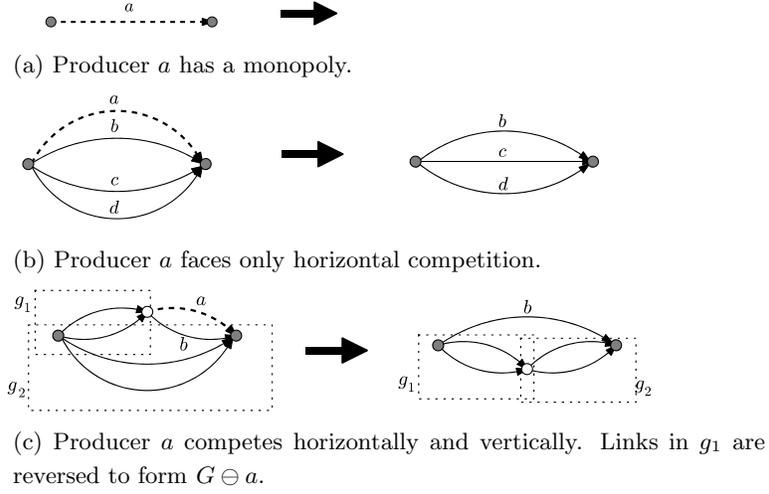


FIGURE 4. The producer's substitute network $G \ominus a$.

no choice but to purchase from a . Since there is no price at which a substitute can be purchased, we say $R_{\ominus a} = \infty$ in this case. In general, $R_{\ominus a}$ measures the market power of producer a in equilibrium, with a higher multiplier indicating a relative absence of attractive alternatives to producer a . The multiplier $R_{\ominus g}$ is defined analogously for any submarket $g \subseteq G$.

The effect of a producer's markup on its own profit is captured through the ratio of w_a to $R_{\ominus a}$. To isolate this effect, we express demand as the product of two factors. One factor depends entirely on this ratio, and the other is a scaling factor, independent of w_a , that measures the demand for producer a when $w_a = 0$. This factor, referred to by $\mu_a(\vec{w}_{-a})$, captures the vertical, rather than horizontal, competition faced by a , and in doing so accounts for the position of a in the market, prior to pivoting.

To simplify notation, we will generally suppress the dependency on the fixed vector \vec{w} . When a faces purely horizontal competition, $\mu_a = 1$ for any value of \vec{w} . In the case of vertical competition, the factor will be strictly less than one, and decreasing in the markups demanded for complements of a . In general, the factor μ_a may be increasing, decreasing, or unaffected by w_b , depending on whether b is largely a substitute or a complement of a . If $\nu_S(a) = (G_1, G_2 \dots G_d)$ is the sequence of series submarkets within which a is nested (these are the even elements of $\nu(a)$), then

$$\mu_a = \prod_{g \in \nu_S(a)} \frac{R_{\ominus g}}{R_{\ominus g} + R_{g \setminus \psi_a(g)}}. \quad (6)$$

For example, in Figure 4c, $\nu_S(a)$ contains a single element, $S(g_1, P(a, b))$, and $\mu_a = R_{g_2} / (R_{g_2} + R_{g_1})$. For a competitor $b \neq a$, there can be at most one level l such that $b \in G_l \setminus \psi_a(G_l)$, indicating a dampening effect of w_b on producer a 's demand, which results because $\psi_a(G_l)$ and $\psi_b(G_l)$ are bundled. An analogous scaling factor, μ_g is defined for $x_g(\vec{w})$ on any submarket $g \subseteq G$.

To summarize, for a fixed vector \vec{w}_{-a} , the parameters μ_a and $R_{\ominus a}$ measure, respectively, the vertical and horizontal competition facing producer a . In Proposition 2, we express $x_a(\vec{w})$ in terms of these two quantities and producer a 's own multiplier. In this formulation, μ_a determines the intercept of producer a 's residual demand, and $R_{\ominus a}$ determines the slope with respect to w_a .

Proposition 2. For a market G with price functions fixed according to \vec{w} , and for any producer a , the equilibrium assignment $\vec{x}(\vec{w})$ takes the form

$$x_a(\vec{w}) = \mu_a \left[\frac{R_{\ominus a}}{R_{\ominus a} + w_a} \right]. \quad (7)$$

Setting the derivative of the profit function with respect to w_a to zero, we can characterize the best response of any producer. The size of producer a 's markup in the first stage will depend on its market power.

Proposition 3. In the markup game, the best response function for any producer a satisfies

$$\Phi_a(\vec{w}_{-a}) = 2c_a + R_{\ominus a}. \quad (8)$$

In terms of $R_{\ominus a}$, the per-unit price that producer a will charge in equilibrium is $p_a(x_a) = w_a x_a = 2c_a x_a + R_{\ominus a} x_a$. Producer a 's costs are given by $\kappa(x_a) = c_a x_a^2$, yielding a marginal cost of $\partial \kappa(x_a) / \partial x_a = 2c_a x_a$. Thus, equilibrium prices can be interpreted intuitively to consist of marginal costs of production, plus a markup of $R_{\ominus a} x_a$. Furthermore, using $R_{\ominus a}$ as a measure of market power, the markup that can be extracted is directly related to the level of competition faced. As the competition faced by producer a increases, $R_{\ominus a}$ will decrease. In the extreme case when $R_{\ominus a}$ tends to zero, the price will approach the marginal cost, in accordance with the interpretation as a competitive market. If $R_{\ominus a}$ becomes small for all producers, then the markup vector approaches $2\vec{c}$, so that the equilibrium assignment approaches x^{OPT} . That is, perfectly competitive markets are efficient.

To further quantify the efficiency of a market, we write the total production costs as a function of the markups. According to Proposition 2:

$$C(\vec{x}(\vec{w})) = \sum_{a \in A_G} c_a \mu_a^2 \left[\frac{R_{\ominus a}}{R_{\ominus a} + w_a} \right]^2. \quad (9)$$

From Proposition 1, we know that x^{OPT} is precisely $\vec{x}(\vec{c})$. Recalling that R_G is the price of any bundle, the optimal production cost is:

$$C(x^{\text{OPT}}) = \sum_{i: B_i \in \mathcal{B}} (R_G |_{\vec{w}=\vec{c}}) f_i = R_G |_{\vec{w}=\vec{c}}. \quad (10)$$

For an arbitrary equilibrium assignment, the total payment by customers is $R_G = \sum_{a \in A_G} w_a [x_a(\vec{w})]^2$. In general, it is not the case that $C(\vec{x}(\vec{w})) = R_G$, since some portion of these payments is kept by the producers as profit. To study efficiency of an assignment $\vec{x}(\vec{w})$, we compare the cost $C(\vec{x}(\vec{w}))$ to $C(x^{\text{OPT}})$. Ultimately, the degree of inefficiency will depend on $\vec{\alpha}$, which is the outcome of the strategic choices taken by producers in the first-stage game.

Example of Markup Equilibrium. To illustrate the concepts put forth in this section, a simple example of competition with markups for a market of substitutes is discussed here.

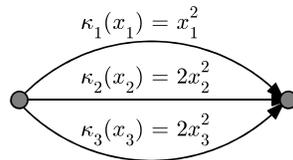


FIGURE 5. Example of a market of substitutes.

The market in Figure 5 consists of three producers of perfect substitutes. Producers 2 and 3 face cost functions $\kappa_a(x_a) = 2x_a^2$, while producer 1 is more efficient, with costs $\kappa_1(x_1) = x_1^2$. Producers

choose markups $\vec{\alpha}$, leading to price functions $p_1(x_1) = \alpha_1 x_1$, $p_2(x_2) = 2\alpha_2 x_2$, and $p_3(x_3) = 2\alpha_3 x_3$ with price multipliers $w_1 = \alpha_1$ and $w_a = 2\alpha_a$ for $a \in \{2, 3\}$. In this setting, $R_{\ominus a} = (\sum_{b \neq a} 1/w_b)^{-1}$ for $a \in \{1, 2, 3\}$. Producers do not compete vertically, so that $\mu_a = 1$ for all a . The assignment for producer a is then $x_a = (1/w_a)(1/w_1 + 1/w_2 + 1/w_3)^{-1}$. In equilibrium, markups satisfy the system:

$$\begin{aligned}\alpha_1 &= \arg \max_{\alpha \geq 0} \left\{ (\alpha - 1) \left(\frac{1/\alpha}{1/\alpha + 1/2\alpha_2 + 1/2\alpha_3} \right)^2 \right\}, \\ \alpha_2 &= \arg \max_{\alpha \geq 0} \left\{ (\alpha - 1) \left(\frac{1/2\alpha}{1/\alpha_1 + 1/2\alpha + 1/2\alpha_3} \right)^2 \right\}, \\ \alpha_3 &= \arg \max_{\alpha \geq 0} \left\{ (\alpha - 1) \left(\frac{1/2\alpha}{1/\alpha_1 + 1/2\alpha_2 + 1/2\alpha} \right)^2 \right\}.\end{aligned}$$

This system can be solved numerically, as is shown in the next section. Solving for the equilibrium gives $\alpha_1 = 5.56$, while $\alpha_2 = \alpha_3 = 3.56$. All products sell at a price of 2.17. This outcome is summarized in Table 1. We note that producer 1 takes advantage of its relative efficiency by charg-

	Pr. 1	Pr. 2	Pr. 3	Market (G)
Efficiency (c)	1	2	2	.5
Markup (α)	5.56	3.56	3.56	4.34
Multiplier (R)	5.56	7.12	7.12	2.17
Market Share (x)	.39	.305	.305	1
Cost (cx^2)	.152	.186	.186	.525

TABLE 1. Equilibrium outcome for the game in Figure 5

ing markups higher than those of its competitors. This makes price functions, as experienced by consumers, more symmetric than the true cost functions. Because of this distortion, less customers purchase from producer 1 in equilibrium than would do so in a socially optimal assignment. The socially optimal assignment x^{OPT} , which solves $\min\{x_1^2 + 2x_2^2 + 2x_3^2 : x_1 + x_2 + x_3 = 1 \text{ and } \vec{x} \geq 0\}$, equals the vector $(1/2, 1/4, 1/4)$ and the optimal social cost is $C(x^{\text{OPT}}) = 1/2$. In comparison, $C(x^{\text{NE}}) = .525 = (1.05)C(x^{\text{OPT}})$. In this case, distortions driven by producer markups lead to a 5% increase in the total costs of production.

3. EQUILIBRIUM OF MARKUP GAME

In this section, \vec{w} refers to an equilibrium of the markup game. Each producer selects w_a to satisfy the best-response map $\Phi_a(\vec{w}_{-a}) = 2c_a + R_{\ominus a}$. A Nash equilibrium is a vector \vec{w} satisfying $w_a = \Phi_a(\vec{w}_{-a})$ for all $a \in A_G$. It is clear from (5) that $R_{\ominus a}(\cdot)$ is a continuous function, and so $\Phi_a(\cdot)$ is continuous and single-valued. Combining the producers' individual best response functions yields a continuous vector-valued function $\Phi(\vec{w})$ whose fixed points, if any exist, correspond to equilibrium markups. If the image of $\Phi(\cdot)$ over the domain $\vec{w} \in \mathbb{R}_+^q$ is contained within $X = \prod_{a \in A_G} X_a$ with $X_a \subset [2c_a, \infty)$, we can, without loss of generality, define markup equilibria as fixed points of the function $\tilde{\Phi} : X \rightarrow X$ where $\tilde{\Phi}_a(\vec{w}) := \Phi_a(\vec{w}_{-a})$. Making use of Brouwer's fixed point theorem, a sufficient condition for existence of a fixed point of $\tilde{\Phi}(\cdot)$ is compactness of X (Fudenberg and Tirole 1991). If producer markups are bounded so that $w_a \leq \bar{w} < \infty$ for all producers, then we define X_a by the compact set $[2c_a, \bar{w}]$, and apply the fixed point theorem. We proceed in this section by deriving conditions which guarantee the existence of \vec{w} . Essentially, an equilibrium requires sufficient competitive pressure to prevent any producer from continually increasing the size of their markup.

3.1. Local Markup Equilibria. If many options exist to produce some offering g , its markup, represented in the aggregate by R_g , will be relatively insensitive to the markups of substitute bundles. Here we formalize the impact of a given submarket g_F on competition in the remaining partial market $G \ominus g_F$. It follows from (8), and the recursion in (5), that the impact of producers in g_F on the other producers is determined entirely by the network price multiplier R_{g_F} . As \vec{w} captures the decisions of all producers in the first-stage game, we let $\vec{w}_{\ominus g_F|g_F}(R_{g_F})$ represent the decisions of producers in $G \setminus g_F$ in a markup game played with R_{g_F} held fixed. As the dynamics of competition within g_F are ignored, we call this vector a *local markup equilibrium*. The proofs in Section 3.3 hinge on the sensitivity of local markups to changes in R_{g_F} . We will show that a sufficient level of redundancy in the structure of G limits this effect so that markups cannot grow too large, and an equilibrium exists. In Section 6, we revisit local markup equilibria in the context of a component producer who assumes constant markups for all but its direct competitors.

For a submarket $g \subseteq G \ominus g_F$, $R_{g|g_F}(\vec{w}_g, R_{g_F})$ is defined analogously to $R_g(\vec{w})$ and computed in the same inductive manner, with the distinction that g_F is considered a leaf of the submarket tree for the local market $G \ominus g_F$. For producer $a \in G \setminus g_F$, the best-response function in the local market game becomes $\Phi_{a|g_F}(\vec{w}_{\{\ominus g_F \setminus a\}|g_F}, R_{g_F}) = 2c_a + R_{\ominus a|g_F}(\vec{w}_{\{\ominus g_F \setminus a\}|g_F}, R_{g_F})$. The first-stage decisions, $\vec{w}_{\ominus g_F|g_F}(R_{g_F})$, then satisfy:

$$w_a = \Phi_{a|g_F}(\vec{w}_{\{\ominus g_F \setminus a\}|g_F}, R_{g_F}) \text{ for all } a \in G \setminus g_F. \quad (11)$$

Clearly, when $R_{g_F} = R_{g_F}(\vec{w}_{g_F})$, the markups selected in the local markup equilibrium match those of the full first-stage game. Furthermore, when $R_{g_F} = \infty$, so that the presence of alternatives outside of $G \ominus g_F$ has no bearing on the market power of firms in the local market, the equilibrium is precisely that resulting from a full game played on $G \ominus g_F$. In general, the distinction between the full game and that corresponding to some finite R_{g_F} is that demand in the local game may be elastic, with a small parameter R_{g_F} indicating the existence of attractive options outside of the market defined by $G \ominus g_F$. Thus, as R_{g_F} shrinks, competition in the local game becomes more intense, and this fact is reflected in the equilibrium vector $\vec{w}_{\ominus g_F|g_F}(R_{g_F})$.

In general, the competitive pressure that a submarket g_F exerts on any disjoint submarket g through the choice of multiplier R_{g_F} is captured by a *submarket response function*,

$$\phi_{g|g_F}(R_{g_F}) := R_{g|g_F}(\vec{w}_{\ominus g_F|g_F}(R_{g_F}), R_{g_F}), \quad (12)$$

whose value reflects the competitive interaction of producers in g , in the context of the local game in which R_{g_F} is held fixed. (For completeness, we note that the same definition may apply when $g_F \subseteq g$, but $\phi_{g|g_F}(\cdot)$ is undefined when g and g_F overlap partially). As defined, $\phi_{g|g_F}(R_{g_F})$ relies on the existence of a unique local equilibrium vector, which is not guaranteed. Conditions for existence of both local and full market equilibria will be developed in the following section.

For convenience, when g_F is not specified, we will interpret $\phi_g(\cdot)$ to depend on a fixed multiplier $R_{\ominus g}$. For the case of a single producer a , $\Phi_a(\vec{w}_{-a}) = \phi_a(R_{\ominus a}) = 2c_a + R_{\ominus a}$, so that $\phi_g(\cdot)$ generalizes the best response function $\Phi_a(\cdot)$ while making the dependence on the substitute network, $G \ominus a$, explicit in the definition. Accordingly, we can redefine a markup equilibrium as a vector \vec{w} such that $w_a = \phi_a(R_{\ominus a})$ for all producers $a \in A_G$.

3.2. Extension to Elastic Demand. The network multiplier R_G is itself the result of a response function $\phi_G(\cdot)$ whose argument is set exogenously, and reflects the price multiplier of an option outside of the market. To this point, by assuming that no alternatives to G exist, we have used implicitly that $R_{\ominus G} = \infty$. By choosing $R_{\ominus G} < \infty$ we allow for an elastic demand. The price of an outside alternative, g_0 , and by extension the willingness to pay for market G , can be defined by a fixed multiplier $R_{\ominus G}$ applied to the function $u(1 - x_G)$ where x_G is the demand assigned to market G . With linear unit costs, the demand takes the form $p_G = R_{\ominus G}(1 - x_G)$, or $x_G = 1 - p_G/R_{\ominus G}$, yielding a model of linear demand and quadratic total costs.

3.3. Graph Connectivity and Existence of Equilibria. In this section, we explore the existence of an upper bound on \vec{w} in markets with inelastic and elastic demand, respectively. In both cases, we will see that the critical property in establishing a bound is the degree of connectivity of the network structure. A set of links whose removal disconnects the graph is a *cut*. A graph is *k-edge-connected* if there are no cuts containing less than k links (Ahuja et al. 1993). For example, Figure 6 shows a 2-connected network. The intuition for studying connectivity was provided in

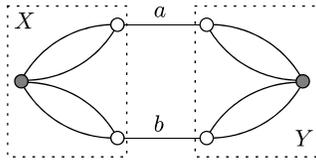


FIGURE 6. A 2-connected network. Producers a and b make up a cut.

Section 2.4, where we noted that a producer's market power is directly related to the uniqueness of the producer's niche, which is reflected in the number (and ultimately, price) of alternative paths available for joining the nodes that the producer connects in G . The connectivity of the graph indicates the smallest set of producers such that one must be used in any path connecting some pair of nodes in the network. A high degree of connectivity should translate to some bound on the market power of any individual producer. In this section we formalize this idea.

For a submarket g , the connectivity $Q(g)$ is the largest k for which g is k -edge-connected. A directed cut is one that divides the graph so that the source and sink are disconnected. A cut that does not separate the source and sink is a *vertical cut* in that the producers in the cut belong to some common bundle and compete vertically. If $Q(g) = k$, then there can be neither any directed cuts, nor any vertical cuts, that contain less than k links. As such, link directions play no role in determining $Q(g)$. Redefining connectivity in terms of vertical cuts alone gives the vertical connectivity $V(g)$. In general, composing g with producers in parallel can increase $Q(g)$, but $V(g)$ provides an upper bound on the connectivity of any market within which g is nested.

It is clear from the response functions that in the case of a duopoly, the combined sensitivity of the producers leads to an infinitely increasing sequence of markups. This applies as well to any network with $Q(G) < 3$. Although G is directed, instability can result from both directed and vertical cuts. Essentially, stability requires that the substitute network for any producer is 2-connected in its directed form. The cause of instability in any case where $G \ominus a$ is not 2-connected is similar to that of the duopoly case, where $G \ominus a$ consists of a single link for either producer. In the general case, it is producer a and the producer that disconnects $G \ominus a$ that combine to drive the instability.

We will show that when the graph is 3-connected, there is enough competition to ensure that markups are bounded. We begin with local markets for which some outside competition exists.

Proposition 4. *For $R_{\ominus g}$ finite, a local markup equilibrium $\vec{w}_{g|\ominus g}(R_{\ominus g})$ exists if and only if $V(g) \geq 3$. In this case, $\vec{w}_{g|\ominus g}(R_{\ominus g})$ is unique and $\phi_g(R_{\ominus g})$ is continuously differentiable.*

The key to establishing existence in G is that producers are arranged into submarkets in such a way that their sensitivity, in the aggregate, to competitors' markups diminishes with the size of those markups. When G is 3-edge-connected, we show, for any producer a , that $\phi'_{\ominus a}(w_a) \rightarrow 0$ as w_a gets large. At equilibrium, w_a is a fixed point of the function $h_a : h_a(w_a) \rightarrow \phi_a(\phi_{\ominus a}(w_a))$. That $h'_a(w_a)$ diminishes for large markups is sufficient to guarantee a finite fixed point, and so establish a finite bound on \vec{w} .

Theorem 1. *A markup equilibrium exists in G if and only if the network is 3-edge-connected.*

By bounding \vec{w} , we restrict the image of $\tilde{\Phi}(\vec{w})$ to a compact set, assuring the existence of a markup equilibrium. We observe further that $\Phi_a(\vec{w}_{-a})$ is increasing in w_b for all $b \neq a$. As a result, any sequence $\{\vec{w}^\tau\}$ with $\vec{w}^\tau = \tilde{\Phi}(\vec{w}^{\tau-1})$ will be increasing element-wise. Starting at \vec{w}^0 with $w_a^0 = 2c_a$ for all $a \in A_G$, we generate a sequence of markups that must converge to a markup equilibrium. Applying iterated best responses, we are able to compute a markup equilibrium in this way for any game that satisfies the 3-connectivity condition.

Corollary 1. *If a markup equilibrium exists, it can be approximated by iterating best responses.*

Remark 1. *Monotonicity of the sequence $\{\vec{w}^\tau\}$ results because markups of all producers are strategic complements. Indeed, in Proposition 9, in the appendix, we show that $\log(\pi_a(w_a, \vec{w}_{-a}))$, has increasing differences in (w_a, \vec{w}_{-a}) . When markups are bounded, the markup game is log-supermodular. (In addition to complementarity, X is then compact and in combination with the standard ordering defines a complete lattice.) This gives an alternative proof of existence via Tarski's fixed point theorem (Topkis 1998).*

Furthermore, we show that the markup equilibrium that we compute is indeed the only equilibrium of the first-stage game. At an equilibrium, \vec{w} , we have $\phi_a(R_{\ominus a})/R_{\ominus a} = w_a/R_{\ominus a} = w_a/\phi_{\ominus a}(w_a)$. In Theorem 2, we show that the first and last ratios in the equality are monotonically decreasing and increasing in $R_{\ominus a}$, respectively. This ensures that an equilibrium can exist for at most one value of $R_{\ominus a}$, and so for at most one value of w_a .

Theorem 2. *If a Nash equilibrium exists in the markup game, then it is unique.*

When demand is elastic, a weaker existence condition is imposed on G . In particular, directed cuts of size $k < 3$, as exist in monopoly or duopoly models, may be present without introducing instability. Using the demand model of Section 3.2 and Proposition 4, it is immediate that an equilibrium exists in an elastic demand market for which the vertical connectivity is at least 3.

Theorem 3. *If the market G is subject to an elastic demand of the form $x_G = 1 - p_G/R_{\ominus G}$, for finite $R_{\ominus G}$, a markup equilibrium exists if and only if $V(G) \geq 3$. When it exists, this equilibrium is unique.*

The conditions of Theorem 3 imply the possibility that an equilibrium does not exist, even in an elastic demand market. The presence of an outside option provides stability when there is a shortage of purchase options, as with a monopoly or duopoly, by assigning some value to not purchasing. If, on the other hand, there is a lack of competition vertically within some bundle, instability will result and persist in the elastic case. Accordingly, the existence of equilibria in an elastic demand market is tied to its vertical connectivity.

For both elastic and inelastic demand markets, $V(G)$ must be at least 3 for an equilibrium to exist. However, if there is some vertical instability, but the directed network from s to t remains 3-connected, there may still be an equilibrium on some subnetwork spanning the market (this holds trivially for an elastic demand market, because the price never exceeds $R_{\ominus G}$). In the next section, we see that in such cases we can simply ignore the paths with vertical instability.

3.4. Irrelevance of Inefficient Submarkets. When the competition in a market is insufficient, then producers will continually have an incentive to increase markups, so that no equilibrium exists. Yet, because we allow for asymmetric market structures, it may happen that some producers face sufficient competition while others do not. Intuitively, if some subnetwork of producers, spanning the market vertically, supports an equilibrium, while other producers raise their markups infinitely, we expect that eventually all customers will abandon the unstable producers and adopt the equilibrium assignment consistent with the stable set of production bundles.

This observation allows us to study some markets that are not 3-connected, but do have a 3-connected substructure embedded within. That is, we consider a market G^+ that is an *extension*

of a 3-connected market G . To ensure that G spans the market vertically, we define an extension as the addition of competition in parallel to a submarket of G . Formally, we form G^+ by replacing some submarket g^* of G with $P(g^*, g^+)$, where g^+ is an SP network.

First, we consider the case where G^+ remains 3-connected, but costs for producers in g^+ are prohibitively large. We take a sequence, indexed by j , of extended networks with costs on each link b in g^+ given by a sequence c_b^j , and assume that $c_b^j \rightarrow \infty$ as $j \rightarrow \infty$. For every j , an equilibrium is induced on G^+ , with an equilibrium price multiplier $R_{g^+}^j$ for g^+ in aggregate, and an equilibrium n -vector \vec{w}^j for producers in G . The point to observe here is that because $R_{g^+}^j$ must grow large with the costs on g^+ , the response functions on G^+ converge to the original response functions as $j \rightarrow \infty$, ensuring that \vec{w} , the equilibrium on G , is recovered in the limit. Thus, g^+ can be ignored when costs are large enough.

Theorem 4. *Let \vec{w} be the unique equilibrium of the markup game on G , and G^+ be a 3-connected extension of G , with $c_b^j \rightarrow \infty$ simultaneously for all $b \in G^+ \setminus G$, when $j \rightarrow \infty$. Then, the sequence $(\vec{w}_{G^+}^j, \vec{w}_{G^+ \setminus G}^j)$ of extended equilibria converges to (\vec{w}, ∞) as $j \rightarrow \infty$.*

In Theorem 4, G^+ is 3-connected, so each instance of the extended network is one that we can analyze on its own. In contrast, we look next at the case where g^+ introduces some instability into the market through its competitive structure. We note that g^+ need not be 3-connected for G^+ to be so. For instance, adding a single link to G cannot reduce the connectivity. However, it may be the case that g^+ is vertically unstable, in that removing two producers from g^+ disconnects g^+ into three disconnected markets. In this case, no extension formed from g^+ will produce an equilibrium. The extended network is no longer 3-connected, and when best-responses are iterated, it must be that $\vec{w}_{G^+ \setminus G}^\tau \rightarrow \infty$, where τ is the number of iterations applied. (Unlike in Theorem 4, costs in the extended network remain constant throughout.) We will show that while the sequence $\{\vec{w}_{G^+ \setminus G}^\tau\}_\tau$ is unbounded, $\vec{w}_G^\tau \rightarrow \vec{w}$, with the limit an equilibrium in G . In this way, we expand the set of networks we may analyze to include all extensions of 3-connected networks.

Theorem 5. *Let \vec{w} be the unique equilibrium of the markup game on G . If $G^+ \ominus G$ is vertically unstable, then multipliers on G converge to \vec{w} when best responses are iterated on G^+ .*

The result supports the use of iterated best responses to analyze any market in which there is an embedded 3-connected network spanning from source to sink. We have defined an extension as the addition of a single submarket, but repeated application of Theorem 5 allows for more general structures.

3.5. General Network Structures. As the following example will demonstrate, Theorem 1 does not immediately generalize to networks that are not series-parallel. Figure 7 presents a very simple network structure that is 3-edge-connected, but violates the restriction to SP structure. No markup equilibrium exists for this network.

Critically, when the network is not SP, we cannot guarantee that all producers are active in equilibrium. In Figure 7, producer 3 is offering a contribution to the bundle that is evidently being offered by producers 1 and 4 as well. Here producer 1 is offering the equivalent of products 2 and 3 in combination. Similarly, producer 4 is offering the equivalent of products 3 and 5 in combination. If the markups and demand allocation are such that the prices for products 1 and 4 are less than the prices of products 2 and 5, respectively, then producer 3 is in effect excluded from the market. There is no markup that producer 3 can choose for which customers will purchase product 3. When this is the case, the price function for product 3 does not influence the second stage results, and as such does not factor into the profits of other producers. Consequently, when producer 3 is not active, we can eliminate it from the analysis entirely, with no affect on the equilibrium. The

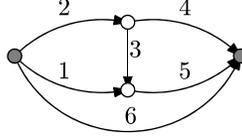


FIGURE 7. A 3-connected network that is not series-parallel.

remaining producers then constitute a series-parallel network that is not 3-connected. There is no equilibrium in such a network, so producer 3 must be active in any equilibrium.

For producer 3 to be active, the price of the bundle $B_4 = \{2, 3, 5\}$, must be equal to that of $B_1 = \{1, 5\}$, $B_2 = \{2, 4\}$, and $B_3 = \{6\}$. For a given set of multipliers \vec{w} , the consumption assignment \vec{f} satisfies:

$$\begin{aligned} f_1 w_1 + (f_1 + f_4) w_5 &= (f_2 + f_4) w_2 + f_2 w_4 = f_3 w_6 = (f_2 + f_4) w_2 + f_4 w_3 + (f_1 + f_4) w_5 \\ f_1 + f_2 + f_3 + f_4 &= 1. \end{aligned} \quad (13)$$

Solving this system for \vec{f} yields the consumption and production assignments for a second-stage equilibrium. After constructing the profit functions for each producer, we find that each producer's optimal markup is again of the form, $\Phi_a(\vec{w}_{-a}) = 2c_a + R_{\ominus a}$, where $R_{\ominus a}$ is the price of an equilibrium assignment in a substitute network.

Structurally, the network in Figure 8 is entirely symmetric, in the sense that $G \ominus a$ has the same structure for any choice of a . The graph of $G \ominus 1$ is pictured in Figure 8, and the logic to follow will apply symmetrically to each producer's markup. For a given set of multipliers \vec{w}_{-1} , we have that

$$R_{\ominus 1} = \begin{cases} \hat{f}_1 w_2 + \hat{f}_1 w_3 = \hat{f}_2 w_5 + \hat{f}_2 w_6 & \text{if } \hat{f}_3 = 0 \\ (\hat{f}_1 + \hat{f}_3) w_2 + \hat{f}_3 w_4 + (\hat{f}_2 + \hat{f}_3) w_5 & \text{if } \hat{f}_3 > 0, \end{cases}$$

where \hat{f} is a consumption assignment satisfying $\hat{f}_1 + \hat{f}_2 + \hat{f}_3 = 1$. If $\hat{f}_3 = 0$, then $R_{\ominus 1} = ((w_2 + w_3)\hat{f}_1 + (w_5 + w_6)\hat{f}_2)/2 \geq \min\{w_2, w_3, w_5, w_6\}$. If $\hat{f}_3 > 0$, then $R_{\ominus 1} \geq w_2 \hat{f}_1 + w_4 \hat{f}_3 + w_5 \hat{f}_2 \geq \min\{w_2, w_4, w_5\}$. Employing the symmetric arguments, $\Phi_a(\vec{w}_{-a}) > \min_{b \in A_G} \{w_b\}$ for all $a \in A_G$, which is a contradiction. It follows that there are no markup equilibria for which producer 3 is active, and consequently, no markup equilibria in the market represented by the 3-connected network, G .

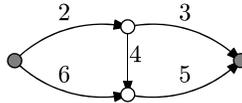


FIGURE 8. Substitute network for producer 1.

4. SENSITIVITY ANALYSIS OF PRODUCER OUTCOMES

In this section, we study the effect of changes to market parameters and structure on the outcomes experienced by a producer a in equilibrium. As observed in Section 2.4, competitor markups can impact a through their effect on $R_{\ominus a}$, which measures horizontal competition, or through μ_a , which measures vertical competition. The impact of any perturbation is understood as a combination of its effects on each of these terms. Here, we analyze these effects for a perturbation of a producer's own efficiency parameter, as well as for changes in the structure of its competition. In the latter

case, we distinguish between competitors positioned vertically to a (whose markups decrease μ_a), and those positioned horizontally (whose markups increase μ_a).

When its competitors become less efficient in their production, producer profits may move in either direction, depending on the relative position of the producer to the altered market. Among our findings, summarized in Table 2, are that:

- (i) an increase in any producer's cost of production increases the markups of *all competitors* in equilibrium.
- (ii) an increase in a producer's own costs *can increase* that producer's equilibrium profits.
- (iii) an increase in the costs of production for complementary items *decreases market share* for efficient producers, but their less efficient competitors may actually *gain share*.

At a high level, the relationships we observe depend on whether the producers in question compete in a fashion that is 'more horizontal' or 'more vertical'.

In general, the changes we consider take the form of a shift in the response function for a single producer or some subnetwork of producers. By a shift, we mean that $\phi_g(R_{\ominus g})$ is replaced by a function $\hat{\phi}_g(R_{\ominus g})$ such that (for an upwards shift) $\hat{\phi}_g(R) \geq \phi_g(R)$ for all R in the domain. For a single producer, a shift in $\phi_a(\cdot)$ can result only from a change in c_a , but for a submarket g it can be the result of any number of structural or parametric changes within g . For example, in Section 5 we show that a merger of horizontal competitors results in an upshift, while a merger that increases vertical integration results in a downshift.

A first observation concerns the sensitivity of price multipliers. As price multipliers of any producers are strategic complements, we are able to show that an upwards (downwards) shift in $\phi_g(\cdot)$ induces an increase (decrease) in the equilibrium multiplier R_g , and in the equilibrium price multipliers of *all* competitors.

Lemma 1. *For any submarkets $g \subseteq G$ and $g' \subseteq G$ satisfying $g \subseteq g'$ or $g' \subseteq G \ominus g$, an upshift in $\phi_g(R_{\ominus g})$ leads to an increase in the equilibrium multipliers R_g and $R_{g'}$.*

The conditions placed on g' in Lemma 1 exclude only subproducts of g , as their behavior is dependent on the nature of the structural change that causes the shift in $\phi_g(\cdot)$. Furthermore, while an upshift in $\phi_g(\cdot)$ raises all competitors' price multipliers, the effect is the greatest for R_g . Corollary 3, in the appendix, implies that the ratio $R_g/R_{g'}$ increases with the shift. In what follows now, we analyze the sensitivity of producer outcomes where vertical competition, as measured by μ_a , also plays a prominent role.

4.1. Own cost perturbation. Consider an increase in c_a , and the corresponding upshift that is brought about in $\phi_a(\cdot)$. In Lemma 1, we show that this drives up w_b for all producers b . For $b \neq a$, this implies an increase in α_b . On the other hand, producer a 's own markup decreases. Among direct competitors, this implies the intuitive result that more efficient producers apply larger markups.

Proposition 5. *For any producer a , α_a decreases in equilibrium when c_a is increased.*

While the percentage markup decreases, the size of the absolute profit margin increases. Increasing c_a by Δ makes w_a increase by $\Delta_a = 2\Delta + \Delta_{\ominus a}$, where $\Delta_{\ominus a}$ is the resulting change in $R_{\ominus a}$. Since $\phi_{\ominus a}(\cdot)$ is unchanged, $R_{\ominus a} = \phi_{\ominus a}(w_a)$ increases, so that $\Delta_a > 2\Delta$. So, for any fixed production quantity x , the profit margin, $(w_a - c_a)x$, increases. On the other hand, the market share, x_a , will decrease following the upshift.

Proposition 6. *For any producer a , x_a decreases in equilibrium when c_a is increased.*

This follows from repeated application of Corollary 3, which indicates that each subproduct containing a applies a proportionally larger increase in its multiplier than do its competitors.

We are most concerned with the change in profits, $\pi_a = (w_a - c_a)x_a^2$. As it turns out, depending on the relative size of the effects on markups and market share, an increase in c_a can result in either an increase or decrease in profits (the numerical example in Table 4 illustrates both cases). This in itself is an intriguing phenomenon, as it suggests that it may in some cases be to a producer's advantage to be *less* efficient. We note that this effect is not unique to models with vertical competition. The critical observation is that for sufficiently inelastic demand, the overall level of profits may be higher in a market where the aggregate cost is larger. Decreasing efficiency weakens a producer's competitive position, but may also increase the overall profits, and this increase may dominate the individual effects. The same effect can occur in a Cournot model, although the cost shock must affect at least 3 producers (Février and Linnemer 2004). In fact, the presence of vertical competitors makes a profit decrease more likely. As c_a increases, the corresponding increase in markups from producers of complementary items contributes to the decrease in producer a 's market share.

Although we observe that profits can shift in either direction, we now show that, regardless of the structure of vertical competition, the price p_a can only increase with c_a .

Proposition 7. *For any producer a , p_a increases in equilibrium when c_a is increased.*

Two responses, the reduction of producer a 's markup and the reallocation of demand to competing producers, both serve to dampen the effect of a cost perturbation. However, some portion of the cost increase is ultimately reflected in the new equilibrium price.

4.2. Competitor perturbation. Also of interest is the impact of the producer's position in relation to a competitor who may alter its efficiency. Here, the impact on profits will depend on the extent to which the producer competes horizontally or vertically with the perturbed producer. There is an aspect of horizontal competition to each producer relationship, with the exact nature captured by the substitute network. (The exception is the case in which two full markets compete serially for a fixed demand, so that producers in one market have no effect on demand for producers in the other). Any increase in a competitor's cost structure will deintensify horizontal competition in the sense that $R_{\ominus a}$ increases. This increases $w_a - c_a$ as a result, and because $\phi'_a(R_{\ominus a}) < w_a/R_{\ominus a}$ (see Corollary 3), x_a/μ_a increases with any competitor's upshift. The result is that π_a/μ_a^2 increases.

In general, π_a can move in either direction, with the determining factor being the direction of the change in μ_a and its size in relation to horizontal effects. With the exception of a producer who spans the market vertically, all producers have some vertical aspect to their relation with competitors. Although, the effect on μ_a can be positive or negative. For more insight, we specialize to a producer a at depth 3 in the submarket tree. The general market of this type is illustrated in Figure 9. The producer faces some local parallel competition, depicted as G_L . In addition, the competition outside of the localized market is divided into vertical and horizontal competition (depicted by the submarkets G_V and G_H , respectively). We allow generality in that G_L , G_V , and G_H can be arbitrary submarkets, but limit the depth of a to 3 for simplicity.

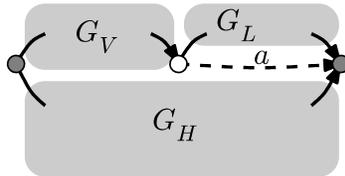


FIGURE 9. General representation of horizontal and vertical competition for a producer at depth 3.

An upshift in $\phi_{G_H}(\cdot)$, positioned horizontally to producer a , leads to an increase in R_{G_H} that is proportionally larger than the increase in R_{G_V} . This implies an increase in μ_a , and thus in π_a .

Proposition 8. *Using the notation of Figure 9, any upshift in $\phi_{G_H}(\cdot)$ increases the equilibrium market share x_a , prices p_a , and profits π_a for producer a .*

Similarly, an upshift in $\phi_{G_V}(\cdot)$, positioned vertically to producer a , leads to a decrease in μ_a . Here, the conclusions in terms of market share and profits for producer a are not clear *a priori*, and will depend on the structure of the individual submarkets. For instance, an efficient producer will lose market share when the cost of complementary items increases (see Proposition 10, in the appendix). However, we observe that the most inefficient producers of a component may stand to gain market share, as this deintensifies local competition. Furthermore, in a market for small-enough components, all producers lose market share when the cost of complements increases, as the vertical effects tend to dominate.

submarket	cost c	price mult. w	markup α	demand x	price p	profit π
a	↑	↑	↓	↓	↑	↓
G_V	-	↑	↑	↓	↓	↓
G_H	-	↑	↑	↑	↑	↑

TABLE 2. Sensitivity analysis of model’s outputs for a cost increase in a submarket. The rows refer to a perturbation of c_a , and for producers in the vertical and horizontal markets, respectively. Arrows point to the possible directions of change of each of the model’s outputs for producer a .

Table 2 summarizes this section’s sensitivity analysis by pointing out the direction of change of each output of the model for producer a . Each row, a , G_V , or G_H , indicates the location of the perturbation. To interpret, the last entry for G_V indicates that, e.g., a CPU manufacturer may gain or lose profits from additional inefficiencies in the market for monitors, and that both outcomes are seen to occur under specific conditions. The last row shows that a cost increase for a producer of integrated computers allows a CPU manufacturer, modeled by a , to increase markups while gaining demand, and so increase its prices and profits. In the case of a producer’s own price, this quantity moves in the same direction as the cost perturbation, although market share, and in some cases profits, move in the opposite direction. Note that the directional changes for perturbations within G_V or G_H apply for any type of upshift.

5. EFFECT OF MERGERS ON MARKET OUTCOMES

Here we will focus on the effect of changes to network structure that leave the overall production capacity fixed. Thus, we see how network structure dictates the intensity of competition, as measured by the effect on total producer profits and the social cost of production. We start in Section 5.1 by looking at the overall markup applied to a bundle. Recalling Lemma 1, structural changes that either intensify or relax competition locally within a particular submarket have the corresponding effect on the overall bundle price, and so we analyze these local effects. In Section 5.2, we observe that the connection between local and global effects does not extend to the social cost criterion. In this case, the full market structure should be considered to assess the impact of a structural change.

For this analysis, recall that R_G is the market price in equilibrium for a given network, and $C(x^{\text{OPT}}) = R_G|_{\vec{w}=\vec{c}}$ is the cost of satisfying demand in a socially optimal manner. A comparison of these terms gives a measure of the extent to which bundles have been marked up. In particular, $R_G/(R_G|_{\vec{w}=\vec{c}})$ measures the ‘average’ markup, and $R_G - R_G|_{\vec{w}=\vec{c}}$ is equal to the total producer profit. In terms of social cost, $C(x^{\text{NE}})$ evaluates an equilibrium markup vector \vec{w} , and the ratio $C(x^{\text{NE}})/(R_G|_{\vec{w}=\vec{c}})$ determines the inefficiency of that vector. As such, we are particularly interested in ‘mergers’; i.e., changes to the market structure for which $R_G|_{\vec{w}=\vec{c}}$ remains constant. In this case, the effects on profits and efficiency can be observed through R_G and $C(x^{\text{NE}})$ alone.

We define a *merger* as a change in the network structure where multiple links are combined in a way that preserves the aggregate cost structure. The cost of the new link should match the cost of using the subnetwork that it replaces, assuming that flow is allocated optimally within the original subnetwork. We denote the optimally aggregated cost of a submarket g by c_g . The procedures for aggregating costs optimally are identical to those for aggregating price multipliers:

$$c_{S(g)} = \sum_{g \in \mathcal{G}} c_g, \quad \text{and} \quad c_{P(g)} = \left(\sum_{g \in \mathcal{G}} 1/c_g \right)^{-1}. \quad (14)$$

In this setting, any changes in market outcomes result entirely from changes in the way the producers interact.

5.1. Industry Markups. Lemma 1 illustrates a consistency between the local competitiveness of subproducts and the markups of the market as a whole. A change in network structure that causes an upshift in a local response function will increase R_G as well. In the case of a merger, this implies an increase in the overall industry markups. In this section, we highlight the local effect of both *horizontal mergers* and *vertical integration*.

5.1.1. Horizontal Mergers. We first look at horizontal mergers, where two parallel links, a_1 and a_2 are combined to form a_P . We denote the cost of the merged producer by c . Letting $\theta = c_{a_2}/(c_{a_1} + c_{a_2})$, we have $c_{a_1} = c/\theta$ and $c_{a_2} = c/(1 - \theta)$. Any horizontal merger can be described in this way for some $\theta \in (0, 1)$. We show that any such merger results in an upwards shifted response function, $\phi_{a_P}(\cdot)$ relative to the aggregated response function $\phi_{g_P}(\cdot)$, where $g_P = P(a_1, a_2)$. This is consistent with intuition since a merger reduces the competition in the market.

Theorem 6. *Horizontal mergers increase the equilibrium price of a bundle.*

With respect to any fixed $R_{\ominus g_P}$, the two producers prior to merging behave equivalently to the elastic duopoly studied in Akgün (2004). There, the equilibrium for the duopoly is derived in closed-form, from which it can be shown that both $\phi_{g_P}(R_{\ominus g_P})$ and $\phi'_{g_P}(R_{\ominus g_P})$ increase as θ gets further from $1/2$. In a series-parallel setting, we can apply Lemma 1 to extend this parametric relationship to the market price R_G (equations (18) and (19) in the appendix show that the result for $\phi'_{g_P}(R_{\ominus g_P})$ can also be extended to larger submarkets). We conclude that not only does any horizontal merger increase the equilibrium price, but the size of the price increase is increasing in the symmetry of the merging firms. That is, for two parallel links with a given aggregate cost, c , the configuration leading to the lowest prices (equivalently, the largest upshift in ϕ_{g_P} upon merging) is that of $c_1 = c_2 = 2c$. Prices increase as the distribution becomes less symmetric, with the highest prices coming from complete asymmetry in which one link is eliminated altogether.

5.1.2. Vertical Integration. We complete our discussion of mergers with the case of mergers involving vertical competitors. Two producers in series is an unstable configuration, so we will not analyze mergers originating from this structure. Rather, we look at the case of a single producer in series with a set of producers who compete with each other horizontally because it is the simplest configuration with vertical competition. We consider the effect of consolidating all of these producers to a single one. We interpret this as a scenario where the production being offered by the parallel competitors is carried out in-house by the producer occupying a single link. In this way we study the effect of *vertical integration*.

Let c be the cost of the integrated producer a_S . We consider parallel producers a_1 and a_2 , comprising a submarket g_P . The submarket g_P is connected in series with a third producer a_V to form $g_S = S(a_V, g_P)$. We require $c_{g_S} = c$, and in particular, for $\theta_P, \theta_V \in (0, 1)$, let $c_{a_1} = (1 - \theta_V)c/\theta_P$, $c_{a_2} = (1 - \theta_V)c/(1 - \theta_P)$, and $c_{a_V} = \theta_V c$. For any choice of θ_P and θ_V , $\phi_{a_S}(\cdot)$ is a downward shift of $\phi_{g_S}(\cdot)$, so that vertical integration results in a lower price than the subcontracting setup.

Theorem 7. *Vertical integration decreases the equilibrium price of a bundle. The size of the effect is decreasing in θ_V .*

The theorem establishes that the markup of an integrated producer a_S provides a lower bound on $\phi_{g_S}(R_{\ominus g_S})$. In the limit as θ_V nears 1, the behavior of g_S resembles that of the integrated producer. The response function then shifts up monotonically as θ_V is decreased. Looking at the competitive portion of the network, g_P , the competition among these producers is most intense when this local market is relatively small. As g_P grows larger relative to its substitutes, the sensitivity $\phi'_{g_P}(R_{\ominus g_P|\ominus g_S})$ to competitor markups, notably those of the monopolist a_V , increases, so that the dynamic of competition within g_S begins to resemble more closely that of serial monopolies. The implication is then that vertical integration produces the largest decrease in bundle price when the local monopolist, a_V , incurs a small portion of the production costs in market g_S . Thus, a lack of competition in, e.g., the market for keyboards, can drive large markups on the price of computers. Though less extreme, the notebook PC example of Dedrick et al. (2010) has Intel’s production costs at a relatively small (roughly 16%) proportion of the total. In the absence of direct competition from AMD, we expect Intel’s strength to have a large effect on PC prices.

5.2. Social Cost of Production. In contrast to industry markups, the social cost of production may decrease following a horizontal merger that consolidates market power locally. Relative to total profit, social cost depends on the symmetry, rather than the size of markups. So, when a bundle consisting of product g is inherently more expensive to produce than substitute bundles, likely leading to high markups on those substitutes, what is perceived locally as an inefficiency in the market for g may be a force that drives markups on g closer to those on substitutes, in effect reducing the degree of distortion in the overall market. We proceed with an example to demonstrate this possible effect.

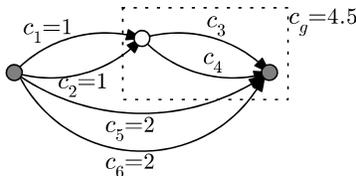


FIGURE 10. Total production cost in this market is smaller when producers 3 and 4 are merged.

Consider the market G in Figure 10, with submarket g . The market for product g is a duopoly, and for a fixed multiplier $R_{\ominus g}$, the two producers face an elastic combined demand. Considering this market for g alone, the most efficient configuration would appear to be the symmetric one (indeed, this is shown for an elastic duopoly model in Akgün (2004)). For comparison, we consider the efficiency that results in G when producers 3 and 4 are merged into a single producer with efficiency parameter $c_g = [1/c_3 + 1/c_4]^{-1}$, so that the aggregate cost structure is maintained.

When costs in g are symmetric, $c_3 = c_4 = 9$. In this case, the optimal allocation has 83.3% of customers purchasing from producers 5 and 6. The average cost of a bundle under the optimum is 0.83. The equilibrium allocation is presented in Table 3. Producers 5 and 6, each being more efficient than the other purchase combinations, apply relatively large markups of $\alpha_5 = \alpha_6 = 5.1$ to their products. In comparison, the price of a combination purchase from the other producers is only 3.1 times larger than the cost of 10. This distortion encourages a larger proportion of costly combination purchases, and the average cost of a bundle in equilibrium is 0.87.

In another scenario, producers 3 and 4 are merged into a single producer with parameter $c_g = 4.5$. Studying g in isolation would suggest that this arrangement is inefficient. Yet, the merged producer

<i>Social Optimum</i>	Pr. 1	Pr. 2	g	Pr. 5	Pr. 6	Market (G)
Efficiency (c)	1	1	4.5	2	2	.833
Market Share (x)	.083	.083	.167	.417	.417	1
Cost (cx^2)	.007	.007	.126	.347	.347	.833

<i>Symmetric Producers</i>	Pr. 1	Pr. 2	Pr. 3	Pr. 4	Pr. 5	Pr. 6	Market (G)
Efficiency (c)	1	1	9	9	2	2	.833
Markup (α)	6.96	6.96	2.70	2.70	5.08	5.08	4.60
Market Share (x)	.123	.123	.123	.123	.377	.377	1
Cost (cx^2)	.015	.015	.135	.135	.285	.285	.870

<i>Merged Producer</i>	Pr. 1	Pr. 2	g	Pr. 5	Pr. 6	Market (G)
Efficiency (c)	1	1	4.5	2	2	.833
Markup (α)	8.12	8.12	4.21	5.90	5.90	5.64
Market Share (x)	.102	.102	.204	.398	.398	1
Cost (cx^2)	.010	.010	.187	.317	.317	.842

TABLE 3. Social cost comparison for symmetric and asymmetric costs in g .

applies a larger markup, that raises the price of a combination purchase to 4.9 times the cost. This shifts some demand back to producers 5 and 6, so that the average cost of a bundle falls to 0.84, despite the market power of the merged producer. Although the difference in social cost between these two scenarios is rather small, the direction of change is surprising as it goes contrary to what a local model of market g suggests. Social cost is discussed further in Appendix B, where we bound the inefficiency of equilibrium production costs in comparison to the optimal allocation of production.

6. EFFECT OF IGNORING VERTICAL COMPETITION

We have directed particular attention towards insights gained from the vertical component of our model. To that end, this section is concerned with model misspecification. We show how misconceptions may form if a substantial vertical aspect is ignored. For instance, consider a CPU manufacturer evaluating a decision to invest in more efficient capacity. This investment will likely trigger smaller markups from producers of other components (i.e. monitors, per the model of Figure 2) and integrated computer models. If the producer restricts analysis to the CPU market, using a *localized* model (as described in Section 3.1), then these markups are implicitly assumed to remain at their pre-investment levels, neglecting the competitors' responses.

Formalizing this concept, for a producer a competing in a parallel submarket g of G , that is $g = P(\mathcal{G})$ with $a \in \mathcal{G}$, an alternative to a full model of G would be to estimate the parameters μ_g and $R_{\ominus g}$, or equivalently to estimate the demand function for the submarket g , and treat g as a market of exclusively horizontal competition, subject to an elastic demand. This approach may be reasonable, and is in line with the way in which producers choose their markups in our model (actually we assume an even more restricted viewpoint in which $R_{\ominus a}$ is held fixed to generate a residual demand). However, a localized approach of this type can yield misleading results with regard to the sensitivity of producer profits to model primitives.

Consider a perturbation of producer a 's own efficiency parameter by Δ , as described in Section 4.1. By altering $\phi'_{G_V}(\cdot)$ and $\phi'_{G_H}(\cdot)$ we demonstrate estimation errors in both directions arising from a localized view (the localized model treats both of these sensitivities as zero). Figure 11 provides two examples; a horizontally sensitive market in (a) and a vertically sensitive market in (b). In both, we fix $c_{G_H} = c_{G_V} = 2$. However, in (a), G_H contains a single link, while G_V is a symmetric pair (note that a single link is more sensitive than any parallel pair), and in (b) we model the reverse. Table 4 summarizes the original equilibrium in each market, as well as the equilibrium when c_a is increased by $\Delta = 1$. Lastly, we look at a localized model, where $R_{\ominus G_{L+}}$ remains fixed to $R_{G_V} + R_{G_H}$ as computed with the original costs, and summarize the equilibrium in this localized model when costs are perturbed.

We observe that the profits estimated by the localized model are too high in the vertically sensitive market and too low in the horizontally sensitive market. Furthermore the example in (a) demonstrates that even the direction of the change in profits may differ between the two models.

These flaws are problematic for a producer evaluating a decision to invest in more efficient technology (i.e. decreasing the efficiency parameter). As such, they provide producers with motivation to explicitly model their vertical competition.



FIGURE 11. A localized model underestimates profits in (a) and overestimates profits in (b).

	Market (a)								Market (b)							
	c_a	w_a	$R_{\ominus a}$	R_V	R_H	μ_a	x_a	π_a	c_a	w_a	$R_{\ominus a}$	R_V	R_H	μ_a	x_a	π_a
Initial	2	13.1	9.1	9.6	20.1	.678	.278	.854	2	13.1	9.1	20.1	9.6	.322	.132	.193
Perturbed	3	16.0	10.0	9.9	21.5	.685	.263	.898	3	16.0	10.0	21.5	9.9	.315	.121	.190
Localized Pert.	3	15.6	9.6	—	—	.678	.258	.842	3	15.6	9.6	—	—	.322	.123	.191

TABLE 4. Summary of perturbed equilibria.

7. CONCLUSION AND FUTURE DIRECTIONS

We have developed a model of competition for producers of substitutes and complements, within the context of the supply function equilibrium paradigm. By restricting to quadratic cost functions, we provide a precise description of producer markups, characterized by a network representation of substitute purchase options for each producer. This intuitive form helps to highlight the relationships between market structure, equilibrium prices, and the cost of production. A number of interesting extensions to this work remain.

We have so far focused on time scales where market structure is considered exogenously, and developed a means to analyze the resulting outcomes. This characterization can be used to evaluate a number of long-term decisions that are outside of the current scope. If strategic selection of network structure is allowed on the part of producers, the model provides a context to study competitive alliances and the effects of bundling complementary items. These decisions amount to restrictions on the set of links that may connect to certain nodes. In the context of assembly supply chains, network design may be carried out centrally by the assembler, with the objective of minimizing the bundle price. We provide insight to these strategic sourcing decisions by modeling the result of competition between the chosen suppliers.

Some technical conditions may be relaxed to provide a closer fit to a greater number of applications. Our analysis of the second-stage assignment will extend in a similar manner to monomial per-unit cost functions of the form, $u_a(x) = c_a x^q$, and an analogous decomposition of horizontal and vertical effects follows. However, for $q \neq 1$, best response functions are not obtained in closed form, but implicitly through a polynomial equation. When an equilibrium exists, markups can be computed iteratively. For $q < 1$, response functions are less sensitive than those studied here, and 3-connectedness remains a sufficient condition for existence. A stronger condition is needed for $q > 1$. For a market of substitutes, Correa et al. (2008) show existence if and only if more than $q+1$ producers compete.

Also of interest is the case where production bundles are imperfect substitutes. Among other phenomena, this will allow for inherent preferences among customers for either integrated or self-assembled product offerings. Heterogeneity of preferences, or an oligopsony setting, in which customers vary in their market power, are other possible directions. For the second-stage game on its own, oligopsonistic outcomes have been considered in, e.g., Cominetti et al. (2009). We hope to enable future work on the dynamics of producer competition in the presence of these and other market considerations.

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APPENDIX A. ROBUSTNESS OF EQUILIBRIA IN MARKUP PRICE FUNCTIONS

We restrict the set of possible price functions that producers may use to the set of *markup price functions*, wherein a scalar multiple is applied to the producer's cost function. A crucial question regarding this modeling assumption is whether the equilibria we model are indeed price function equilibria in a more general sense. In the following, we answer that question in the affirmative.

Theorem 8. *Assume that each producer $i \in A$ bids a markup price function of the form $p_i(x_i) = \alpha_i u_i(x_i)$ for an $\alpha_i > 0$ of their choice. If these are at equilibrium in the space of markup price functions, then they are at equilibrium in the space of all non-decreasing price functions.*

Proof. We need to show for any producer $a \in A_G$, that $p_a(\cdot)$ is a best response, among all non-decreasing price functions, to the price functions chosen by the other producers. To do this, we hold the price function $p_i(\cdot)$ fixed for $i \neq a$, and focus on the producer a . With all other functions fixed, we will show that for the entire set of supply functions such that the price that a charges in equilibrium is p , there is a unique value of x_a that producer a will produce in equilibrium. As this value depends only on p , we denote it by the mapping $\gamma(p)$. We will see that $\gamma(p)$ is a decreasing function defined on the domain $[0, \bar{p}]$, where \bar{p} is the price such that $\gamma(p) = 0$. For $p > \bar{p}$, producer a does not produce in any equilibrium.

To define $\gamma(\cdot)$, consider the sequence $\nu(a)$ of submarkets within which a is nested. We will show that for any $g \in \nu(a)$, x_g is a strictly increasing function of x_a . Note that for any submarket g' not containing a , all producers have chosen markup price functions, and thus the price in g' is fixed to some continuous, strictly increasing function $p_{g'}(x_{g'})$. Now, consider any parallel submarket $g \in \nu(a)$. Then $g = P(\psi_a(g), g')$, where $g' = g \setminus \psi_a(g)$. Assume that both $x_{\psi_a(g)}$ and $p_{\psi_a(g)}$ are determined uniquely by x_a , with $x_{\psi_a(g)}$ strictly increasing, and $p_{\psi_a(g)}$ non-decreasing in x_a . Then $p_g = p_{\psi_a(g)}$ and since $p_{g'} = p_{\psi_a(g)}$, and $p_{g'}(x_{g'})$ is strictly increasing, $x_g = x_{\psi_a(g)} + x_{g'} = x_{\psi_a(g)} + p_{g'}^{-1}(p_{\psi_a(g)})$. Thus, p_g is nondecreasing in x_a , and x_g is strictly increasing in x_a , with both determined uniquely. For a series submarket $g \in \nu(a)$, we have $g = S(\psi_a(g), g')$, where $g' = g \setminus \psi_a(g)$. We apply the same assumptions on $x_{\psi_a(g)}$ and $p_{\psi_a(g)}$. Here, $x_g = x_{g'} = x_{\psi_a(g)}$ and $p_g = p_{\psi_a(g)}(x_{\psi_a(g)}) + p_{g'}(x_{\psi_a(g)})$. Again, p_g is nondecreasing in x_a , and x_g is strictly increasing in x_a , with both determined uniquely. Finally, we observe that p_a is fixed to p , and x_a is trivially considered a strictly increasing function of x_a , allowing us to begin induction with $g = \psi_a^{-1}(a)$.

We conclude by induction on the submarket tree that x_G is strictly increasing in x_a . Now, because $x_G = 1$, there is only a single value of x_a that is consistent with our chosen p . We call this value $\gamma(p)$. Furthermore, if $\psi_a^{-1}(a)$ is parallel, then $x_{\psi_a^{-1}(a)}$ increases strictly with p , for any fixed x_a . If $\psi_a^{-1}(a)$ is a series submarket, then $p_{\psi_a^{-1}(a)}$ increases strictly in p , and consequently $x_{\psi_a^{-2}(a)}$ increases strictly in p , for an fixed x_a . In either case, a larger p leads to a larger x_g for some g in ν_a , and this shift propagates up the submarket tree, so that x_G increases strictly in p for any x_a . As a result, $\gamma(p)$ is strictly decreasing in p , for $p < \bar{p}$.

Excluding the redundant choices $p > \bar{p}$, it is evident that the set $\{(p, \gamma(p)) : p \in [0, \bar{p}]\}$ contains all price-quantity pairs that producer a can possibly achieve in an equilibrium. Furthermore, the equilibrium outcome of any nondecreasing price function chosen by a is determined only by the point at which the function's inverse crosses the graph of $\gamma(p)$. Therefore, each of these pairs is achieved by all non-decreasing price functions that cross the point it describes. As such, producer a 's best response problem amounts to solving the maximization problem

$$\max_{p \in [0, \bar{p}]} \{\gamma(p)p - \kappa_a(\gamma(p))\} \tag{15}$$

and then selecting any price function that goes through the optimal price-quantity pair. Because the shape of the price function is irrelevant, there is always a markup price function, $p_a^*(x_a)$ that is

a best response. Since $p_a(\cdot)$ is the best choice among markup price functions, it must be that $p_a(\cdot)$ is equal to $p_a^*(\cdot)$, and so $p_a(\cdot)$ is a best response in the larger set as well. \square

APPENDIX B. INEFFICIENCY BOUND

Let $\vec{\alpha}$ be the unique markup equilibrium for an arbitrary 3-edge-connected market structure. Let the scalar $\bar{\alpha}$ be the upper bound on markups for that market, which is guaranteed to exist when \bar{w} exists. Then:

$$2C(x^{\text{NE}}) = 2 \sum_{a \in A_G} c_a [x_a^{\text{NE}}]^2 \leq \sum_{a \in A_G} \alpha_a c_a [x_a^{\text{NE}}]^2 = R_G \leq \bar{\alpha} R_G|_{\bar{w}=\vec{c}} = \bar{\alpha} C(x^{\text{OPT}}),$$

with the first inequality following from (8), so that inefficiency can be bounded as follows:

$$\frac{C(x^{\text{NE}})}{C(x^{\text{OPT}})} \leq \frac{\bar{\alpha}}{2}. \quad (16)$$

Here we take $\bar{\alpha}$, the upper bound on producer markups, to represent $\max_a \{\alpha_a\}$, making the inequality as tight as possible. For this bound to be meaningful, we would like to express, or at least bound, $\bar{\alpha}$ as a function of the model primitives. To do this, we introduce the term σ_a , which is an indicator of producer a 's market power. We define $\sigma_a := (R_{\ominus a}|_{\bar{w}=\vec{c}})/c_a$. We then define $\sigma := \max_a \{\sigma_a\}$, as a measure of asymmetries in the network as a whole.

From (8), $\alpha_a = 2 + R_{\ominus a}/c_a \leq 2 + \sigma \bar{\alpha}$. This implies that $\bar{\alpha} \leq 2 + \sigma \bar{\alpha}$, from which we deduce that $\bar{\alpha} \leq 2/(1 - \sigma)$ for $\sigma < 1$. Plugging into (16) establishes:

$$\frac{C(x^{\text{NE}})}{C(x^{\text{OPT}})} \leq \frac{1}{1 - \sigma}. \quad (17)$$

We cannot use this bound for $\sigma \geq 1$. Clearly, it is not tight for σ close to 1 as well, as the right-hand side blows up for σ approaching 1 from below. If the market is very competitive, then σ will be close to zero, at which point the bound approaches 1. When general series-parallel markets are considered, σ can be typically made large by introducing additional vertical competition through link subdivisions. Some structural restrictions are thus necessary to guarantee any level of efficiency.

We note that slackness is introduced in our bound by the maximum in the definition of σ . This is necessary due to the possible asymmetry of our market structure. When producers compete horizontally in a single market, this analysis can produce a much tighter bound, due to the fact that the degree of competition faced by each of the producers is closely linked in that type of setting (Correa et al. 2008). Other structural symmetries can be imposed to produce a similar effect.

APPENDIX C. EXISTENCE AND UNIQUENESS PROOFS FROM SECTION 3.3 (INCLUDES SUPPORTING DEFINITIONS)

The approach in this section is to characterize submarkets based on their connectivity, by defining the submarket type $T(g)$, and based on properties of their local equilibria, by defining the submarket type $E(g)$. Two lemmas, Lemma 2 and Lemma 3, are introduced to establish an equivalence between these two characterizations. This provides the underlying relationship between connectivity and equilibrium properties needed for the results presented in Section 3.3. Proof of these results is given once the necessary preliminaries have been established.

Definition [Connectivity Properties]. *We can express connectivity in terms of the operations $S(\cdot)$ and $P(\cdot)$. Let the connectivity of a submarket g be denoted $Q(g)$. The vertical connectivity,*

$V(g)$, is the maximum connectivity of any market containing g . It is the number of edge removals needed to disconnect g into three sections. The following properties define $Q(g)$ and $V(g)$:

$$V(a) = \infty, Q(a) = 1 \text{ for all } a$$

$$V(S(\mathcal{G})) = \min\left\{\min_{g', g'' \in \mathcal{G}, g' \neq g''} \{Q(g') + Q(g'')\}, \min_{g' \in \mathcal{G}} \{V(g')\}\right\}$$

$$Q(S(\mathcal{G})) = \min_{g' \in \mathcal{G}} \{Q(g')\}$$

$$V(P(\mathcal{G})) = \min_{g' \in \mathcal{G}} \{V(g')\}$$

$$Q(P(\mathcal{G})) = \min\left\{\sum_{g' \in \mathcal{G}} \{Q(g')\}, V(P(\mathcal{G}))\right\}.$$

We are concerned, in particular, with three classes of submarkets, characterized by the notation $T(g)$. If $V(g) < 3$, which cannot be the case for a submarket of a 3-connected market, we say $T(g) = 0$. Among markets with $V(g) \geq 3$, we distinguish between those, type 1 submarkets, for which $Q(g) = 1$, and those, type 2 submarkets, for which $Q(g) \geq 2$. Any individual producer constitutes a type 1 submarket. Then $T(g)$ develops as:

$$\begin{aligned} T(S(\mathcal{G})) &= 0 \quad \text{if } \min_{g', g'' \in \mathcal{G}, g' \neq g''} \{T(g') + T(g'')\} \leq 2 \\ &= \min_{g' \in \mathcal{G}} \{T(g')\} \quad \text{otherwise,} \\ T(P(\mathcal{G})) &= 0 \quad \text{if } \min_{g' \in \mathcal{G}} \{T(g')\} = 0 \\ &= 2 \quad \text{otherwise.} \end{aligned}$$

Definition [Boundedness Properties]. For a submarket g such that $\vec{w}_g(R_{\ominus g})$ exists, we say that g is ρ -bounded if there exists $\rho < 1$, and constants K_1 and K_2 such that:

$$\phi_g(R_{\ominus g}) \leq K_1 + K_2 R_{\ominus g}^\rho \text{ for all } R_{\ominus g} < \infty.$$

Similarly, g is 1-bounded if there exists $\rho < 1$, and constants K_1 and K_2 such that:

$$\phi_g(R_{\ominus g}) \leq K_1 + K_2 R_{\ominus g}^\rho + R_{\ominus g} \text{ for all } R_{\ominus g} < \infty.$$

Definition [Local Equilibrium Properties]. We can characterize submarkets in terms of the existence and boundedness of their local equilibria by introducing the following submarket types, represented by $E(g)$:

$$E(g) = 0 \Leftrightarrow \vec{w}_{g|\ominus g}(R_{\ominus g}) \text{ does not exist for } R_{\ominus g} \in (0, \infty]$$

$$E(g) = 1 \Leftrightarrow \vec{w}_{g|\ominus g}(R_{\ominus g}) \text{ exists and is unique for } R_{\ominus g} < \infty,$$

$$\phi_g(R_{\ominus g}) \text{ is continuously differentiable, with } \phi'_g(R_{\ominus g}) < \frac{\phi_g(R_{\ominus g})}{R_{\ominus g}} \text{ for } R_{\ominus g} < \infty,$$

$$\text{and } g \text{ is 1-bounded, with } \frac{\phi_g(R_{\ominus g})}{R_{\ominus g}} > 1.$$

$$E(g) = 2 \Leftrightarrow \vec{w}_{g|\ominus g}(R_{\ominus g}) \text{ exists and is unique for } R_{\ominus g} < \infty,$$

$$\phi_g(R_{\ominus g}) \text{ is continuously differentiable, with } \phi'_g(R_{\ominus g}) < \frac{\phi_g(R_{\ominus g})}{R_{\ominus g}} \text{ for } R_{\ominus g} < \infty,$$

$$\text{and } g \text{ is } \rho\text{-bounded.}$$

Note that the conditions for $E(g) = 1$ and $E(g) = 2$ assure that $\phi_g(R_{\ominus g})/R_{\ominus g}$ is strictly decreasing in $R_{\ominus g}$.

Lemma 2. *The type, $E(S(\mathcal{G}))$, of a series composition is defined inductively as:*

$$\begin{aligned} E(S(\mathcal{G})) &= 0 \quad \text{if } \min_{g', g'' \in \mathcal{G}, g' \neq g''} \{E(g') + E(g'')\} \leq 2 \\ &= \min_{g' \in \mathcal{G}} \{E(g')\} \quad \text{otherwise.} \end{aligned}$$

Proof. Let $g = S(\mathcal{G})$. For the first statement, the result is immediate when $\min_{g' \in \mathcal{G}} \{E(g')\} = 0$, since no equilibrium can exist on g if no local equilibria exist on some component. Furthermore, if there are components g' and g'' with $E(g') = 1$ and $E(g'') = 1$, then $\phi_{g'}(R_{\ominus g'|\ominus g}) > R_{\ominus g'|\ominus g} > R_{g''|\ominus g}$ and $\phi_{g''}(R_{\ominus g''|\ominus g}) > R_{\ominus g''|\ominus g} > R_{g'|\ominus g}$, so no equilibrium can exist.

For the second statement, we assume that $E(g') \in \{1, 2\}$ for all $g' \in \mathcal{G}$, and $E(g') = 1$ for at most one component market. Then we must show that $E(g)$ is equal to the smallest type among its component markets, be it type 1 or type 2. All other components markets are type 2, and beginning with the market of smallest type, we can add the other markets to the composition sequentially (rather than as a single composition of perhaps more than 2 components), so that it is sufficient to show $E(S(g', g'')) = E(g')$ when $E(g') \in \{1, 2\}$ and $E(g'') = 2$. Let $R_{\ominus g}$ be equal to a fixed finite multiplier, r .

Existence. For series composition, $R_{\ominus g'|\ominus g} = r + R_{g''|\ominus g}$ and $R_{\ominus g''|\ominus g} = r + R_{g'|\ominus g}$. Since local equilibria exist for g' and g'' when $R_{\ominus g'|\ominus g}$ and $R_{\ominus g''|\ominus g}$ are finite, respectively, $\vec{w}_{g|\ominus g}(r)$ fails to exist only if both $R_{g'|\ominus g}$ and $R_{g''|\ominus g}$ are unbounded. At a local equilibrium for g , $R_{g'|\ominus g}$ is a fixed point of $h_{g'|\ominus g} : h_{g'|\ominus g}(R) \rightarrow \phi_{g'}(r + \phi_{g''}(r + R))$, where $h_{g'|\ominus g}(0) > 0$ and $h_{g'|\ominus g}(\cdot)$ is continuous. Then, $h'_{g'|\ominus g}(R) = \phi'_{g'}(r + \phi_{g''}(r + R))\phi'_{g''}(r + R) < [\phi_{g'}(r + \phi_{g''}(r + R))/(r + \phi_{g''}(r + R))][\phi_{g''}(r + R)/(r + R)]$. Looking at this upper bound, the first fraction is bounded above by $\phi_{g'}(r + \phi_{g''}(r))/(r + \phi_{g''}(r))$, and the second fraction approaches 0 for large R since g'' is ρ -bounded. Thus, there is some finite point \hat{R} , such that $h'_{g'|\ominus g}(\hat{R}) < 1 - \epsilon$ for $\epsilon \in (0, 1)$. A fixed point of $h_{g'|\ominus g}$ must exist and be less than $\hat{R}/(1 - \epsilon)$.

Uniqueness. Furthermore, $h'_{g'|\ominus g}(R) < 1$ at and to the right of any fixed point, so that the fixed point must be unique. The uniqueness of $R_{g'|\ominus g}$ implies the uniqueness of $R_{g''|\ominus g}$ and $\vec{w}_{g|\ominus g}(r)$ because both $\vec{w}_{g'|\ominus g}(R_{\ominus g'})$ and $\vec{w}_{g''|\ominus g}(R_{\ominus g''})$ are unique by assumption.

C¹ response. Both $\phi_{g'|\ominus g}(\cdot)$ and $\phi_{g''|\ominus g}(\cdot)$ are continuous, since $h_{g'|\ominus g}(R_{g'|\ominus g})$ varies continuously with the parameter r . The derivatives, $\phi'_{g'|\ominus g}(\cdot)$ and $\phi'_{g''|\ominus g}(\cdot)$ can be expressed in terms of the continuous functions $\phi'_{g'}(\cdot)$ and $\phi'_{g''}(\cdot)$. Let $\delta_{g'} = \phi'_{g'}(r + \phi_{g''|\ominus g}(r))$ and $\delta_{g''} = \phi'_{g''}(r + \phi_{g'|\ominus g}(r))$. We observe that $\phi'_{g'|\ominus g}(r) = \delta_{g'}[1 + \phi'_{g''|\ominus g}(r)]$ and $\phi'_{g''|\ominus g}(r) = \delta_{g''}[1 + \phi'_{g'|\ominus g}(r)]$. Solving this system gives:

$$\phi'_{g'|\ominus g}(r) = \delta_{g'}(1 + \delta_{g''})(1 - \delta_{g'}\delta_{g''})^{-1} \quad (18)$$

and the symmetric expression for $\phi'_{g''|\ominus g}(r)$. Note that when $\vec{w}_{g|\ominus g}(r)$ exists, $\delta_{g'}\delta_{g''} < [\phi_{g'}(r + \phi_{g''|\ominus g}(r))/(r + \phi_{g''|\ominus g}(r))][\phi_{g''}(r + \phi_{g'|\ominus g}(r))/(r + \phi_{g'|\ominus g}(r))] = [R_{g'|\ominus g}/(r + R_{g''|\ominus g})][R_{g''|\ominus g}/(r + R_{g'|\ominus g})] < 1$. Finally, we combine to get $\phi'_g(r) = \phi'_{g'|\ominus g}(r) + \phi'_{g''|\ominus g}(r)$, which is single-valued and continuous in r .

Bounded derivative. By assumption, $\phi'_{g'|\ominus g}(r) < \phi_{g'|\ominus g}(r)/r$ and $\phi'_{g''|\ominus g}(r) < \phi_{g''|\ominus g}(r)/r$. Then $\phi'_g(r) = \phi'_{g'|\ominus g}(r) + \phi'_{g''|\ominus g}(r) < \phi_g(r)/r$.

Bounded response. By assumption, $R_{g'|\ominus g} = \phi_{g'}(R_{\ominus g'|\ominus g}) \leq K_1 + K_2 R_{\ominus g'|\ominus g}^\rho + R_{\ominus g'|\ominus g} = K_1 + K_2(r + R_{g''|\ominus g})^\rho + r + R_{g''|\ominus g}$ and $R_{g''|\ominus g} = \phi_{g''}(R_{\ominus g''|\ominus g}) \leq K_3 + K_4 R_{\ominus g''|\ominus g}^q = K_3 + K_4(r + R_{g'|\ominus g})^q$

for some choice of constants. Thus:

$$R_{g''|\Theta g} \leq K_3 + K_4 r^q + K_4 R_{g'|\Theta g}^q \leq (K_3 + K_4 K_1^q) + (K_4 + K_4 K_2^q + K_4) r^q + (K_4 K_2^q + K_4) R_{g''|\Theta g}^q$$

From the above, either $R_{g''|\Theta g} \leq (2K_4 K_2^q + 2K_4)^{\left(\frac{1}{1-q}\right)}$ OR $R_{g''|\Theta g} \leq 2[(K_3 + K_4 K_1^q) + (K_4 + K_4 K_2^q + K_4) r^q]$.

In either case $R_{g''|\Theta g} \leq K'_3 + K'_4 r^q$, where $K'_3 := \max\{(2K_4 K_2^q + 2K_4)^{\left(\frac{1}{1-q}\right)}, 2(K_3 + K_4 K_1^q)\}$ and $K'_4 := 2(K_4 + K_4 K_2^q + K_4)$.

If $E(g') = 1$, then $R_{g'|\Theta g} \leq K_1 + K_2(r + R_{g''|\Theta g})^\rho + r + R_{g''|\Theta g} \leq K'_1 + K'_2 r^{\bar{\rho}} + r$ where $K'_1 := K_1 + K_2(K'_3)^\rho + K'_3$, $K'_2 := K_2 + K_2(K'_4)^\rho + K'_4$, and $\bar{\rho} = \max\{\rho, q\}$. If $E(g') = 2$, then $R_{g'|\Theta g} \leq K_1 + K_2(r + R_{g''|\Theta g})^\rho \leq K''_1 + K''_2 r^{\bar{\rho}}$ where $K''_1 := K_1 + K_2(K'_3)^\rho$ and $K''_2 := K_2 + K_2(K'_4)^\rho$. Since $\phi_g(r) = R_{g'|\Theta g} + R_{g''|\Theta g}$, g is 1-bounded if and only if g' is 1-bounded, and g is ρ -bounded if and only if g' is ρ -bounded.

Lastly, If $E(g') = 1$, then $\phi_g(r)/r > \phi_{g'|\Theta g}(r)/r = \phi_{g'}(r + \phi_{g''|\Theta g}(r))/r > \phi_{g'}(r + \phi_{g''|\Theta g}(r))/(r + \phi_{g''|\Theta g}(r)) > 1$. \square

Lemma 3. *The type, $E(P(\mathcal{G}))$, of a parallel composition is defined inductively as:*

$$\begin{aligned} E(P(\mathcal{G})) &= 0 \quad \text{if } \min_{g' \in \mathcal{G}} \{E(g')\} = 0 \\ &= 2 \quad \text{otherwise.} \end{aligned}$$

Proof. The first statement follows immediately since no equilibrium can exist on $P(\mathcal{G})$ if no local equilibria exist on some component. For the second statement, let $g = P(\mathcal{G})$. Because $E(g) = 1$ is a weaker assumption than $E(g) = 2$, and composition can be done sequentially, it is sufficient that $E(P(\mathcal{G})) = 2$ when $\mathcal{G} = \{g', g''\}$ with $E(g') = 1$ and $E(g'') = 1$. Let $R_{\Theta g}$ be equal to a fixed finite multiplier, r .

Existence. For parallel composition, $R_{\Theta g'|\Theta g}$ and $R_{\Theta g''|\Theta g}$ are bounded above by r . Then, local equilibria exist for both g' and g'' corresponding to any pair $R_{g'|\Theta g}$, $R_{\Theta g''|\Theta g}$ with $R_{g'|\Theta g} = \phi_{g'}(R_{\Theta g'|\Theta g})$ and $R_{\Theta g''|\Theta g} = \phi_{g''}(R_{\Theta g''|\Theta g})$. Equivalently, $R_{g'|\Theta g}$ must be a fixed point of $h_{g'|\Theta g} : h_{g'|\Theta g}(R) \rightarrow \phi_{g'}([1/\phi_{g''}([1/R + 1/r]^{-1}) + 1/r]^{-1})$, where $h_{g'|\Theta g}(0) > 0$ and $h_{g'|\Theta g}(\cdot)$ is continuous. Seeing as $R_{g'|\Theta g} \in [0, \phi_{g'}(r)]$, such a fixed point must exist, by Brouwer's fixed point theorem.

Uniqueness. Furthermore, see that $\partial \phi_{g'}([1/R + 1/r]^{-1})/\partial R = \phi'_{g'}([1/R + 1/r]^{-1})[r/(r + R)]^2 < U_{g'}(R)$, where $U_{g'}(R) := \phi_{g'}([1/R + 1/r]^{-1})[r/(r + R)]R^{-1}$, and the analogous bound, $U_{g''}(R)$ holds for g'' . Then $h'_{g'|\Theta g}(R) < U_{g'}(\phi_{g''}([1/R + 1/r]^{-1}))U_{g''}(R)$, which is decreasing in R , and for a fixed point $R_{g'|\Theta g}$ with $R_{\Theta g''|\Theta g} = \phi_{g''}([1/R_{g'|\Theta g} + 1/r]^{-1})$, $h'_{g'|\Theta g}(R_{g'|\Theta g}) < U_{g'}(R_{\Theta g''|\Theta g})U_{g''}(R_{g'|\Theta g}) = [r/(r + R_{g''|\Theta g})][r/(r + R_{g'|\Theta g})] < 1$. Then $h'_{g'|\Theta g}(R) < 1$ at and to the right of any fixed point, so that the fixed point must be unique. The uniqueness of $R_{g'|\Theta g}$ implies the uniqueness of $R_{\Theta g''|\Theta g}$ and $\vec{w}_{g|\Theta g}(r)$ because both $\vec{w}_{g'|\Theta g'}(R_{\Theta g'})$ and $\vec{w}_{g''|\Theta g''}(R_{\Theta g''})$ are unique by assumption.

C¹ response. Both $\phi_{g'|\Theta g}(\cdot)$ and $\phi_{g''|\Theta g}(\cdot)$ are continuous, since $h_{g'|\Theta g}(R_{g'|\Theta g})$ varies continuously with the parameter r . The derivatives, $\phi'_{g'|\Theta g}(\cdot)$ and $\phi'_{g''|\Theta g}(\cdot)$ can be expressed in terms of the continuous functions $\phi'_{g'}(\cdot)$ and $\phi'_{g''}(\cdot)$. Let $\delta_{g'} = \phi'_{g'}([1/\phi_{g''|\Theta g}(r) + 1/r]^{-1})$ and $\delta_{g''} = \phi'_{g''}([1/\phi_{g'|\Theta g}(r) + 1/r]^{-1})$. We observe that $\phi'_{g'|\Theta g}(r) = \delta_{g'} \phi_{\Theta g'|\Theta g}^2(r) [\phi'_{g''|\Theta g}(r)/\phi_{g''|\Theta g}^2(r) + 1/r^2]$,

along with symmetric expression for g'' . Solving this system gives:

$$\phi'_{g'|\Theta g}(r) = \frac{\delta_{g'} \left[(\phi_{g''|\Theta g}(r)/(\phi_{g''|\Theta g}(r) + r))^2 + \delta_{g''} (r/(\phi_{g''|\Theta g}(r) + r))^2 (\phi_{g'|\Theta g}(r)/(\phi_{g'|\Theta g}(r) + r))^2 \right]}{\left[1 - \delta_{g'} \delta_{g''} (r/(\phi_{g'|\Theta g}(r) + r))^2 (r/(\phi_{g''|\Theta g}(r) + r))^2 \right]} \quad (19)$$

where again the symmetric expression for $\phi'_{g''|\Theta g}(r)$ holds. Combining the two gives $\phi'_g(r) = \phi_g(r)^2 [\phi'_{g'|\Theta g}(r)/\phi_{g'|\Theta g}^2(r) + \phi'_{g''|\Theta g}(r)/\phi_{g''|\Theta g}^2(r)]$ which is single-valued and continuous in r .

Bounded derivative. By assumption, $\phi'_{g'|\Theta g}(r) < \phi_{g'|\Theta g}(r)/r$ and $\phi'_{g''|\Theta g}(r) < \phi_{g''|\Theta g}(r)/r$. Then $\phi'_g(r) = \phi_g(r)^2 [\phi'_{g'|\Theta g}(r)/\phi_{g'|\Theta g}^2(r) + \phi'_{g''|\Theta g}(r)/\phi_{g''|\Theta g}^2(r)] < \phi_g(r)/r$.

Bounded response. By assumption, $R_{g'|\Theta g} = \phi_{g'|\Theta g}(R_{g'|\Theta g}) \leq K_1 + K_2 R_{g'|\Theta g}^\rho + R_{g'|\Theta g} = K_1 + K_2 (1/r + 1/R_{g'|\Theta g})^{-\rho} + (1/r + 1/R_{g'|\Theta g})^{-1}$ and $R_{g''|\Theta g} = \phi_{g''|\Theta g}(R_{g''|\Theta g}) \leq K_3 + K_4 R_{g''|\Theta g}^q + R_{g''|\Theta g} = K_3 + K_4 (1/r + 1/R_{g''|\Theta g})^{-q} + (1/r + 1/R_{g''|\Theta g})^{-1}$. Thus, letting $\bar{K}_1 := \max\{K_1, K_3\}$, $\bar{K}'_2 := \max\{K_2, K_4\}$, $\bar{\rho} := \max\{\rho, q\}$ and $\bar{R} = \max\{R_{g'|\Theta g}, R_{g''|\Theta g}\}$, we get:

$$\bar{R} \leq \bar{K}_1 + \bar{K}'_2 (1/r + 1/\bar{R})^{-\bar{\rho}} + (1/r + 1/\bar{R})^{-1}$$

Or equivalently:

$$\bar{R}^2 \leq \bar{K}_1 \bar{R} + \bar{K}_1 r + \bar{K}'_2 \bar{R} r^{\bar{\rho}} + \bar{K}'_2 \bar{R}^{\bar{\rho}} r$$

From here we consider two cases. First, if $\bar{R} < \sqrt{r}$, then $\phi_g(r) \leq \frac{1}{2}\bar{R} < \frac{1}{2}r^{\frac{1}{2}}$ so g is ρ -bounded. Otherwise, $\bar{R}^{\bar{\rho}} r \leq \bar{R}^{\bar{\rho}} r (\bar{R} r^{-\frac{1}{2}})^{(1-\bar{\rho})} = \bar{R} r^{\frac{1+\bar{\rho}}{2}}$, so that $\bar{R}^2 \leq \bar{K}_1 \bar{R} + \bar{K}_1 r + 2\bar{K}'_2 \bar{R} r^{\frac{1+\bar{\rho}}{2}}$. This implies $\bar{R} \leq R^*$ where $(R^*)^2 - (\bar{K}_1 + 2\bar{K}'_2 r^{\frac{1+\bar{\rho}}{2}})R^* - \bar{K}_1 r = 0$. Solving the quadratic gives:

$$R^* = \frac{1}{2} \left(\bar{K}_1 + 2\bar{K}'_2 r^{\frac{1+\bar{\rho}}{2}} + \sqrt{(\bar{K}_1 + 2\bar{K}'_2 r^{\frac{1+\bar{\rho}}{2}})^2 + 4\bar{K}_1 r} \right) \leq \bar{K}_1 + 2\bar{K}'_2 r^{\frac{1+\bar{\rho}}{2}} + \sqrt{\bar{K}_1 r}$$

In this case, $\phi_g(r) \leq \frac{1}{2}R^* \leq \frac{1}{2}\bar{K}_1 + \left(\frac{1}{2}\sqrt{\bar{K}_1} + \bar{K}_2\right) r^{\frac{1+\bar{\rho}}{2}}$, so g is ρ -bounded. \square

Proof of Proposition 4. We first note that $V(g) \geq 3$ if and only if $T(g) \in \{1, 2\}$. The proof proceeds by showing that $E(g) = T(g)$ for all submarkets g . The results of the Proposition hold for g such that $E(g) \in \{1, 2\}$, so this equivalence will be sufficient. First, observe that $E(a) = T(a)$ for all producers a . In particular, $T(a) = 1$ for all $a \in A_G$. Furthermore, a unique local equilibrium $w_a = 2c_a + R_{\Theta a}$ clearly exists, and $\phi_a(R_{\Theta a}) = 2c_a + R_{\Theta a}$ satisfies all conditions for $E(a) = 1$. Having shown this, we can then extend the equivalence to all submarkets $g \subseteq G$ by induction on the submarket tree, using Lemma 2 and Lemma 3 for series and parallel compositions, respectively. \square

Corollary 2. *Let g be any submarket of a 3-edge-connected market G . If $Q(g) \geq 2$, then $\phi_g(R_{\Theta g})/R_{\Theta g} \rightarrow 0$ as $R_{\Theta g} \rightarrow \infty$.*

Proof. It follows by induction, using Lemma 2 and Lemma 3 that $E(g') = T(g')$ for all submarkets $g' \subseteq G$. It must be that $V(g) \geq 3$, so that $T(g) = 2$. It follows from the definition of $E(\cdot)$ that g is ρ -bounded. The corollary then follows. \square

Corollary 3. *Let g_F be a submarket of G such that $V(G \ominus g_F) \geq 3$. Then for any submarket g such that either $g \subseteq G \ominus g_F$ or $g_F \subseteq g$, the response function $\phi_{g|g_F}(\cdot)$ satisfies $\phi'_{g|g_F}(R_{g_F}) < \phi_{g|g_F}(R_{g_F})/(R_{g_F})$. Consequently, $\phi_{g|g_F}(R_{g_F})/R_{g_F}$ is decreasing in R_{g_F} .*

Proof. It follows by induction, using Lemma 2 and Lemma 3 that $E(g') = T(g')$ for all submarkets $g' \subseteq G$. It follows from the definition of $E(\cdot)$ that $\phi'_{g'}(R_{\ominus g'}) < \phi_{g'}(R_{\ominus g'})/R_{\ominus g'}$. Now, let g be composed of two components submarkets g_I and g_O , and hold R_{g_I} fixed to induce $R_{g_O|g_I} = \phi_{g_O|g_I}(R_{g_I})$ and $R_{\ominus g|g_I} = \phi_{\ominus g|g_I}(R_{g_I})$. From (18) and (19), $\phi'_{g_O|g_I}(R_{g_I}) < R_{g_O|g_I}/R_{g_I}$ when g is series or parallel, respectively. Here, $\phi'_{g_I|g_I}(R_{g_I}) = 1 = R_{g_I}/R_{g_I}$. It follows that $\phi'_{g|g_I}(R_{g_I}) < \phi_{g|g_I}(R_{g_I})/(R_{g_I})$. For g_F nested deeper within g ,

$$\begin{aligned}\phi'_{g|g_F}(R_{g_F}) &= \phi'_{g|\psi_{g_F}(g)}(R_{\psi_{g_F}(g)|g_F})\phi'_{\psi_{g_F}(g)|\psi_{g_F}^2(g)}(R_{\psi_{g_F}^2(g)|g_F}) \cdots \phi'_{\psi^{-1}(g_F)|g_F}(R_{g_F}) \\ &< (R_{g|g_F}/R_{\psi_{g_F}(g)|g_F}) \cdots (R_{\psi^{-1}(g_F)|g_F}/R_{g_F}) \\ &= \phi_{g|g_F}(R_{g_F})/(R_{g_F}).\end{aligned}$$

Then, for $g \subseteq G \ominus g_F$, disjoint from g_F ,

$$\begin{aligned}\phi'_{g|g_F}(R_{g_F}) &= \phi'_g(R_{\ominus g|g_F})\phi'_{\ominus g|g_F}(R_{g_F}) \\ &< (R_{g|g_F}/R_{\ominus g|g_F})(R_{\ominus g|g_F}/R_{g_F}) \\ &= \phi_{g|g_F}(R_{g_F})/(R_{g_F}).\end{aligned}$$

□

Proof of Theorem 1. Consider a markup equilibrium given by a fixed point of the function $\tilde{\Phi}$. If the underlying graph is not 3-edge-connected, then by definition there exists a pair of links, a and b , such that removing a and b disconnects the graph. In the case where removal of a single edge a disconnects the graph, then $R_{\ominus a} = \infty$, so the relation $\Phi_a(\vec{w}_{-a}) = 2 + R_{\ominus a}/c_a$, ensures there can be no fixed point. If the graph is 2-edge-connected, there is a cut consisting of two producers a and b . When removing these two links, the graph is divided into sections, g and g' . (One or both of these may be empty. In the latter case a and b are a duopoly). The substitute network for producer a is $S(g, b, g')$, so that producer b itself defines a cut in $G_{\ominus a}$, and the reverse holds for $G_{\ominus b}$. Then, $R_{\ominus a} = R_g + w_b + R_{g'}$ and $R_{\ominus b} = R_g + w_a + R_{g'}$, so that any equilibrium must satisfy $w_a > R_{\ominus a} \geq w_b$ and $w_b > R_{\ominus b} \geq w_a$. This is a contradiction, so no equilibrium can exist for a graph that is not 3-edge-connected.

If the network is 3-edge-connected then $Q(G \ominus a) \geq 2$ for all $a \in A_G$. Now, making use of Corollary 2 and Corollary 3, $\phi'_{\ominus a}(w_a) \rightarrow 0$ as $w_a \rightarrow \infty$ for any producer a . The best response for producer a in equilibrium is a fixed point of the mapping $h_a : h_a(w_a) \rightarrow \phi_a(\phi_{\ominus a}(w_a))$. $h_a(0) > 0$ and $h_a(\cdot)$ is continuous. Because $\phi'_a(R_{\ominus a}) = 1$, there is some finite point \hat{w} , such that $h'_a(\hat{w}) < 1 - \epsilon$ for $\epsilon \in (0, 1)$. The unique fixed point of h must exist and be less than $\hat{w}/(1 - \epsilon)$. Thus w_a is bounded by a constant for each producer a . By setting $\bar{w} = \max_{a \in A_G} w_a$ we can restrict \vec{w} to the compact region $\prod_{a \in A_G} [2c_a, \bar{w}]$. By Brouwer's fixed point theorem, a markup equilibrium exists. □

Proposition 9. For each producer a , the log of producer profit function, $\log(\pi_a(w_a, \vec{w}_{-a}))$, has increasing differences in (w_a, \vec{w}_{-a}) .

Proof. The log profit function is $\log(\pi_a(w_a, \vec{w}_{-a})) = \log(w_a - c_a) + 2 \log(x_a(\vec{w}))$. Then, $\partial \log(\pi_a(\vec{w}))/\partial w_a = 1/(w_a - c_a) + 2(\partial x_a/\partial w_a)/(x_a(\vec{w})) = 1/(w_a - c_a) - 1/(w_a + R_{\ominus a})$. For any producer $b \neq a$, $R_{\ominus a}$ is increasing in w_b , so that $\partial^2 \log(\pi_a(\vec{w})) / (\partial w_a \partial w_b) > 0$. □

Proof of Theorem 2. For any producer a , w_a is a fixed point of the mapping $h_a : h_a(w_a) \rightarrow \phi_a(\phi_{\ominus a}(w_a))$. Then, $h'_a(w_a) = \phi'_a(\phi_{\ominus a}(w_a))\phi'_{\ominus a}(w_a) < [\phi_a(\phi_{\ominus a}(w_a))/\phi_{\ominus a}(w_a)][\phi_{\ominus a}(w_a)/w_a]$. Both fractions in the upper bound are decreasing in w_a , and their product is equal to 1 at a fixed point

of $h_a(\cdot)$. Thus, $h'_a(w_a) < 1$ at, and to the right of, any fixed point. Therefore, $h_a(w_a) = w_a$ can be satisfied by at most one point. Since this holds for all $a \in A_G$, \vec{w} is unique when it exists. \square

Proof of Theorem 3. The theorem follows directly from Proposition 4. \square

APPENDIX D. OTHER PROOFS

Proof of Proposition 1. As $C(x)$ is convex, and using the first-order conditions, x^{OPT} is the unique assignment for which there exists a consumption assignment f^{OPT} with $(x^{OPT}, f^{OPT}) \in \mathcal{F}$ and

$$\sum_{a \in B_i} 2c_a x_a^{OPT} \leq \sum_{b \in B_j} 2c_b x_b^{OPT} \quad (20)$$

for all $B_i, B_j \in \mathcal{B}$ such that $f_i^{OPT} > 0$. Recalling Equation (4), we see that when all producers use the same markup, the condition in (20) is necessary for x^{NE} . It follows that $x^{OPT} = \vec{x}(k\vec{c})$ for any scalar $k > 0$. \square

Proof of Proposition 2. We will extend this property inductively to all submarkets, including individual producers, beginning with the full market G . Because demand is inelastic, $x_G = 1$. The factor $\mu_G = 1$, and for any finite R_G , the property (7) holds since $R_{\ominus G} = \infty$. Now for a submarket g' , we assume that (7) holds for $g = \psi^{-1}(g')$. If g is composed in series, then $g = S(g', g'')$ where $g'' = g \setminus g'$. Then $R_g = R_{g'} + R_{g''}$ and $R_{\ominus g'} = R_{g''} + R_{\ominus g}$. Since g is series, we adjust the scaling factor so that $\mu_{g'} = \mu_g R_{\ominus g} / (R_{\ominus g} + R_{g''})$. Since $x_{g'}(\vec{w}) = x_g(\vec{w})$:

$$\begin{aligned} x_{g'}(\vec{w}) &= \mu_g [1 + R_g/R_{\ominus g}]^{-1} = \mu_g [1 + R_{g'}/R_{\ominus g} + R_{g''}/R_{\ominus g}]^{-1} \\ &= \mu_g [1 + R_{g''}/R_{\ominus g}]^{-1} [1 + R_{g'}/(R_{\ominus g} + R_{g''})]^{-1} = \mu_{g'} [1 + R_{g'}/R_{\ominus g'}]^{-1}. \end{aligned}$$

If g is composed in parallel, then $g = P(g', g'')$ where $g'' = g \setminus g'$. Then $R_g = [1/R_{g'} + 1/R_{g''}]^{-1}$ and $R_{\ominus g'} = [1/R_{\ominus g} + 1/R_{g''}]^{-1}$. Since g is parallel, $\mu_{g'}$ is exactly μ_g and $x_{g'}(\vec{w}) = x_g(\vec{w})R_g/R_{g'}$. So:

$$\begin{aligned} x_{g'}(\vec{w}) &= (R_g/R_{g'}) x_g(\vec{w}) = (R_g/R_{g'}) \mu_g [1 + R_g/R_{\ominus g}]^{-1} = (R_g/R_{g'}) \mu_{g'} [1 + (1/R_{\ominus g'} - 1/R_{g''}) R_g]^{-1} \\ &= \mu_{g'} [R_{g'}/R_g - R_{g'}/R_{g''} + R_{g'}/R_{\ominus g'}]^{-1} = \mu_{g'} [1 + R_{g'}/R_{\ominus g'}]^{-1}. \end{aligned}$$

The result holds by induction on the submarket tree. \square

Proof of Proposition 3. Having shown from the second-stage game that $x_a(\vec{w}) = \mu_a [R_{\ominus a}/(w_a + R_{\ominus a})]$, in the first stage, producer a chooses $w_a \geq 1$ that maximizes $\pi_a(w_a, \vec{w}_{-a})$. As $w_a \rightarrow c_a$, then $\pi_a(w_a, \vec{w}_{-a}) \rightarrow 0$ and as $w_a \rightarrow \infty$, then $\pi_a(w_a, \vec{w}_{-a}) = O(w_a^{-1}) \rightarrow 0$. Evidently, the profit-maximizing markup is interior in (c_a, ∞) . From the first-order optimality conditions, $[x_a(\vec{w})]^2 + 2(w_a - c_a)[x_a(\vec{w})][\partial x_a(\vec{w})/\partial w_a] = 0$, and thus $w_a = c_a - \frac{1}{2}[x_a(\vec{w})][\partial x_a(\vec{w})/\partial w_a]^{-1}$. Since $R_{\ominus a}$ and μ_a do not depend on w_a , it is straightforward to differentiate $x_a(\vec{w})$, getting $\partial x_a(\vec{w})/\partial w_a = -x_a(\vec{w})/(R_{\ominus a} + w_a)$. Note that this term is nonzero for any finite \vec{w} . Substituting into the above we get: $w_a = c_a + \frac{1}{2}(R_{\ominus a} + w_a)$. \square

Proof of Theorem 4. To specify from among the sequence $\{\vec{w}^j\}$ of equilibria, we reintroduce the explicit dependence of $R_g(\vec{w}_g)$ on \vec{w}_g . The equilibrium \vec{w} in G uniquely satisfies $w_a = \phi_a(R_{G \ominus a}(\vec{w}_{G \setminus a}))$ for all $a \in A_G$. These conditions can be restated as $w_a = \phi_{a|g^*}(R_{g^*}(\vec{w}_{g^*}))$ for all $a \in A_{G/g^*}$ and $w_b = \phi_{b|G \ominus g^*}(R_{G \ominus g^*}(\vec{w}_{G/g^*}))$ for all $b \in g^*$. In this vein, we define $H : \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n$

such that $H_a(\vec{w}) = w_a - \phi_{a|g^*}(R_{g^*}(\vec{w}_{g^*}))$ for $a \in A_{G/g^*}$ and $H_b(\vec{w}) = w_b - \phi_{b|G \ominus g^*}(R_{G \ominus g^*}(\vec{w}_{G/g^*}))$ for $b \in g^*$. The equilibrium \vec{w} uniquely satisfies $H(\vec{w}) = \vec{0}$.

A necessary condition for the equilibrium \vec{w}^j is $w_a^j = \phi_{a|g^*}^j(R_{g^*}(\vec{w}_{g^*}^j))$ for all $a \in A_{G/g^*}$ and $w_b^j = \phi_{b|G \ominus g^*}^j(R_{G \ominus g^*}(\vec{w}_{G/g^*}^j))$ for all $b \in g^*$, where, in terms of the response functions from the original markup game in G , $\phi_{a|g^*}^j(R_{g^*}) := \phi_{a|g^*}((1/R_{g^*} + 1/R_{g^+}^j)^{-1})$ and $\phi_{b|G \ominus g^*}^j(R_{G \ominus g^*}) := \phi_{b|G \ominus g^*}((1/R_{G \ominus g^*} + 1/R_{g^+}^j)^{-1})$. We define $H^+ : \mathbb{R}_+^{n+1} \rightarrow \mathbb{R}_+^n$ such that $H_a^+(\vec{w}, R_{g^+}) := w_a - \phi_{a|g^*}((1/R_{g^*}(\vec{w}_{g^*}) + 1/R_{g^+})^{-1})$ for $a \in A_{G/g^*}$ and $H_b^+(\vec{w}, R_{g^+}) := w_b - \phi_{b|G \ominus g^*}((1/R_{G \ominus g^*}(\vec{w}_{G/g^*}) + 1/R_{g^+})^{-1})$ for $b \in g^*$, and thus $H^+(\vec{w}^j, R_{g^+}^j) = 0$ for all j . The continuity of response functions implies that, $H^+(\vec{w}_G, R_{g^+}^j) \rightarrow H(\vec{w})$ as $j \rightarrow \infty$, for any vector \vec{w} .

For $b \in G^+/G$, $w_b^j \geq c_b^j$, so $w_b^j \rightarrow \infty$. Since $H^+(\cdot)$ is continuous,

$$H(\lim_{j \rightarrow \infty} \vec{w}_G^j) = H^+(\lim_{j \rightarrow \infty} \vec{w}_G^j, \lim_{j \rightarrow \infty} R_{g^+}^j) = \lim_{j \rightarrow \infty} H^+(\vec{w}_G^j, R_{g^+}^j) = 0$$

The unique solution to $H(\lim_j \vec{w}_G^j) = 0$ is $\lim_j \vec{w}_G^j = \vec{w}$. \square

Proof of Theorem 5. To specify from among the sequence $\{\vec{w}^\tau\}$ of equilibria, we reintroduce the explicit dependence of $R_g(\vec{w}_g)$ on \vec{w}_g . Let $\{R_{g^+}^\tau\}_\tau$ be the infinite sequence of multipliers induced on $G^+ \ominus G$ when iterating best responses on G^+ . Taking this sequence as fixed, we can simulate the evolution of \vec{w}_G^τ , by the dynamics:

$$w_a^{\tau+1} = \phi_{a|g^*}^\tau(R_{g^*}(\vec{w}_{g^*}^\tau)) := \phi_{a|g^*}((1/R_{g^*}(\vec{w}_{g^*}^\tau) + 1/R_{g^+}^\tau)^{-1}) \text{ for all } a \in A_{G/g^*}$$

$$w_b^{\tau+1} = \phi_{b|G \ominus g^*}^\tau(R_{G \ominus g^*}(\vec{w}_{G/g^*}^\tau)) := \phi_{b|G \ominus g^*}((1/R_{G \ominus g^*}(\vec{w}_{G/g^*}^\tau) + 1/R_{g^+}^\tau)^{-1}) \text{ for all } b \in A_g^*$$

This determines an increasing sequence of vectors $\{\vec{w}_G^\tau\}_\tau$, beginning with $\vec{w}_G^0 = \vec{c}_G$. Looking at $a \in A_{G/g^*}$ we observe that $\phi_{a|g^*}((1/R_{g^*}(\vec{w}_{g^*}^\tau) + 1/R_{g^+}^\tau)^{-1})$ is increasing in both $\vec{w}_{g^*}^\tau$ and $R_{g^+}^\tau$, so that:

$$\phi_{a|g^*}^\tau(R_{g^*}(\vec{w}_{g^*}^\tau)) \leq \lim_{j \rightarrow \infty} \phi_{a|g^*}^j(R_{g^*}(\vec{w}_{g^*}^\tau)) = \phi_{a|g^*}(R_{g^*}(\vec{w}_{g^*}^\tau))$$

So, given that $\vec{w}_G^\tau \leq \vec{w}$,

$$w_a^{\tau+1} = \phi_{a|g^*}^\tau(R_{g^*}(\vec{w}_{g^*}^\tau)) \leq \phi_{a|g^*}(R_{g^*}(\vec{w}_{g^*}^\tau)) \leq \phi_{a|g^*}(R_{g^*}(\vec{w}_{g^*})) = w_a$$

The analogous argument holds for $b \in A_g^*$. Since $\vec{w}^0 = \vec{c} \leq \vec{w}$, we see by induction that $\{\vec{w}_G^\tau\}_\tau$ is bounded above by \vec{w} . Because $\{\vec{w}_G^\tau\}_\tau$ is increasing and bounded, it must converge to a limit \vec{w} . Furthermore, the limit must satisfy:

$$\bar{w}_a = \lim_{\tau \rightarrow \infty} \phi_{a|g^*}^\tau(R_{g^*}(\bar{w}_{g^*})) = \phi_{a|g^*}(R_{g^*}(\bar{w}_{g^*})) \text{ for all } a \in A_{G/g^*}$$

$$\bar{w}_b = \lim_{\tau \rightarrow \infty} \phi_{b|G \ominus g^*}^\tau(R_{G \ominus g^*}(\bar{w}_{G/g^*})) = \phi_{b|G \ominus g^*}(R_{G \ominus g^*}(\bar{w}_{G/g^*})) \text{ for all } b \in A_g^*$$

The unique solution to this system is $\bar{w} = \vec{w}$. \square

Proof of Lemma 1. Let $\bar{g} = \psi_g^{-1}(g)$ be a composition of the submarkets g and g'' . We will show that an upshift in $\phi_g(R_{\ominus g})$ implies upshifts in $\phi_{\bar{g}}(R_{\ominus \bar{g}})$ and in $\phi_{g''|_{\ominus \bar{g}}}(R_{\ominus \bar{g}})$. The result then follows by induction up the submarket tree. $\phi_{\bar{g}}(R_{\ominus \bar{g}})$ is an increasing function of $R_{g|_{\ominus \bar{g}}}$ and $R_{g''|_{\ominus \bar{g}}}$. If \bar{g} is series then $R_{g''|_{\ominus \bar{g}}} = \phi_{g''|_{\ominus \bar{g}}}(R_{\ominus \bar{g}}) = \phi_{g''}(R_{\ominus \bar{g}} + R_{g|_{\ominus \bar{g}}})$, and $R_{g|_{\ominus \bar{g}}}$ is the unique fixed point of $h_{g|_{\ominus \bar{g}}} : h_{g|_{\ominus \bar{g}}}(R) \rightarrow \phi_g(R_{\ominus \bar{g}} + \phi_{g''}(R_{\ominus \bar{g}} + R))$. If \bar{g} is parallel then $R_{g''|_{\ominus \bar{g}}} = \phi_{g''|_{\ominus \bar{g}}}(R_{\ominus \bar{g}}) = \phi_{g''}([1/R_{\ominus \bar{g}} + 1/R_{g|_{\ominus \bar{g}}}]^{-1})$, and $R_{g|_{\ominus \bar{g}}}$ is the unique fixed point of $h_{g|_{\ominus \bar{g}}} : h_{g|_{\ominus \bar{g}}}(R) \rightarrow \phi_g([1/R_{\ominus \bar{g}} + 1/\phi_{g''}([1/R_{\ominus \bar{g}} + 1/R]]^{-1})]^{-1})$. In either case, $R_{g''|_{\ominus \bar{g}}}$ is increasing in $R_{g|_{\ominus \bar{g}}}$. Furthermore,

the inner function of the composition $h_{g|\ominus\bar{g}}$ is unaffected by the shift in $\phi_g(\cdot)$, while the outer function is shifted upwards. Thus, $h_{g|\ominus\bar{g}}(R)$ is shifted upwards. The function $h_{g|\ominus\bar{g}}(R)$ is continuous and $h_{g|\ominus\bar{g}}(0) > 0$ so that $h_{g|\ominus\bar{g}}(R)$ intersects with the 45 degree line once and from above. The upwards shift pushes this point to a larger value of R , thus increasing $R_{g|\ominus\bar{g}}$ for any fixed value of $R_{\ominus\bar{g}}$. As a result, $R_{g''|\ominus\bar{g}}$ and $\phi_{\bar{g}}(R_{\ominus\bar{g}})$ increase as well. \square

Proof of Proposition 5. In equilibrium, w_a is a fixed point of $h_a : h_a(w_a) \rightarrow \phi_a(\phi_{\ominus a}(w_a))$. Perturbing c_a by a small Δ increases w_a by $\Delta_w = 2\Delta/(1 - h'_a(w_a)) = 2\Delta/(1 - \phi'_{\ominus a}(R_{\ominus a}))$. By Corollary 3, $\phi'_{\ominus a}(R_{\ominus a}) < R_{-a}/w_a = 1 - 2/\alpha_a$, and so $\Delta_w < \alpha_a\Delta$. The effect on α_a , denoted Δ_α , satisfies $\Delta_w = c_a\Delta_\alpha + \alpha_a\Delta$ and so $\Delta_\alpha < 0$. \square

Proof of Proposition 6. Recall that x_a can be written as $\prod_{g \in \nu_P(a)} R_g/R_{\psi_a(g)}$. Each term $R_g/R_{\psi_a(g)}$ in this product is equivalently written as $R_{g \setminus \psi_a(g)}/(R_{\psi_a(g)} + R_{g \setminus \psi_a(g)})$. For any $g \in \nu_P(a)$, $R_{\psi_a(g)}$ increases with c_a by Lemma 1. We note that $\phi_{g \setminus \psi_a(g)}(\cdot)$ remains unchanged, and applying Corollary 3, $\phi'_{\{g \setminus \psi_a(g)\}|\psi_a(g)}(R_{\psi_a(g)}) < R_{g \setminus \psi_a(g)}/R_{\psi_a(g)}$, so that $R_{g \setminus \psi_a(g)}/R_{\psi_a(g)}$ decreases. This is true for each $g \in \nu_P(a)$, causing $R_g/R_{\psi_a(g)}$ to decrease, and so x_a decreases. \square

Proof of Proposition 7. Recall that x_a can be written as $\prod_{g \in \nu_P(a)} R_g/R_{\psi_a(g)}$, where $\nu_P(a) = (G_1, G_2, \dots, G_d)$ is the sequence of parallel submarkets within which producer a is nested. We rewrite this product as $x_a = R_{G_1} \left[\prod_{l \in \{2 \dots d\}} R_{G_l}/R_{\psi_a^{-1}(G_l)} \right] R_{\psi_a^{-1}(G_d)}$. R_{G_1} increases with c_a by Lemma 1. Applying Lemma 1 and Corollary 3 for each term in the brackets ensures this product is increasing as well, leaving only $R_{\psi_a^{-1}(G_d)}$ to decrease with c_a . Now looking at $p_a = w_a x_a$, we need only show that $w_a/R_{\psi_a(G_d)}$ is nondecreasing with c_a . If a has direct competition, i.e. $a \in \psi(G_d)$, then this term cancels out of p_a . Otherwise, $w_a/R_{\psi_a(G_d)} = w_a/R_{\psi_a^{-1}(a)}$ is increasing in c_a by Corollary 3. \square

Proof of Proposition 8. By Lemma 1, R_{G_H} increases with the upshift in $\phi_{G_H}(\cdot)$. Considering that $G_H \subseteq R_{\ominus a}$ and $\mu_a = R_{G_H}/(R_{G_V} + R_{G_H})$, Corollary 3 implies an increase in both μ_a and $x_a/\mu_a = R_{\ominus a}/(R_{\ominus a} + w_a)$, so that x_a increases. Lemma 1 also implies an increase in w_a and $w_a - c_a$, so that p_a and π_a increase as well. \square

Proposition 10. *Using the notation of Figure 9, when c_a is small enough, an upshift in $\phi_{G_V}(\cdot)$ decreases the equilibrium market share x_a .*

Proof. Define a sequence $\{c_a^j\}$ of efficiency parameters for producer a , and the corresponding sequence $\{\vec{w}^j\}$ of equilibria. To specify from among the sequence $\{\vec{w}^j\}$ of equilibria, we reintroduce the explicit dependence of $R_g(\vec{w}_g)$ on \vec{w}_g . Let $\{c_a^j\}$ be such that $c_a^j \rightarrow 0$ as $j \rightarrow \infty$. We note that the response functions $\phi_{G_V}(\cdot)$, $\phi_{G_H}(\cdot)$, and $\phi_{G_L}(\cdot)$ are unaffected by c_a^j , so that the equilibrium \vec{w}^j is dictated entirely by the function $\phi_a^j(R_{\ominus a}) = 2c_a^j + R_{\ominus a}$. Furthermore, the smoothness of response functions, and of $\phi_a^j(R_{\ominus a})$ with respect to c_a^j , ensures that $\{\vec{w}^j\} \rightarrow \vec{w}$ as $j \rightarrow \infty$, where \vec{w} is the equilibrium corresponding to $\phi_a(R_{\ominus a}) := R_{\ominus a}$, and that $\phi_g^j(R_g(\vec{w}_g^j)) \rightarrow \phi_g'(R_g(\vec{w}_g))$ for $g \subseteq G$. As a result, letting Δ_x^j represent the change in $x_a(\vec{w}^j)$ resulting from a particular upshift in $\phi_{G_V}(\cdot)$, we see that $\Delta_x^j \rightarrow \Delta_x$, where Δ_x is the change in x_a when $c_a = 0$. When $c_a = 0$, producer a 's market share is given by $x_a(\vec{w}) = \mu_a(\vec{w})R_{\ominus a}(\vec{w})/(R_{\ominus a}(\vec{w}) + w_a) = \mu_a(\vec{w})/2$. Applying Corollary 3 to $\mu_a(\vec{w}) = R_{G_H}(\vec{w})/(R_{G_H}(\vec{w}) + R_{G_V}(\vec{w}))$, we see that x_a decreases with a shift in $\phi_{G_V}(\cdot)$, and that $\Delta_x < 0$. So, for c_a near enough to zero, producer a loses market share. \square

Proof of Theorem 6. We first look at the pre-merger response function for g_P . Here, $\phi_{g_P}(R_{\ominus g_P}) = [1/w_{a_1|\ominus g_P} + 1/w_{a_2|\ominus g_P}]^{-1}$, where:

$$\begin{aligned} w_{a_1|\ominus g_P} &= \phi_{a_1|\ominus g_P}(R_{\ominus g_P}) = \phi_{a_1}(R_{\ominus a_1|\ominus g_P}) = 2c/\theta + [1/R_{\ominus g_P} + 1/w_{a_2|\ominus g_P}]^{-1} \\ w_{a_2|\ominus g_P} &= \phi_{a_2|\ominus g_P}(R_{\ominus g_P}) = \phi_{a_2}(R_{\ominus a_2|\ominus g_P}) = 2c/(1-\theta) + [1/R_{\ominus g_P} + 1/w_{a_1|\ominus g_P}]^{-1} \end{aligned}$$

We observe that $w_{a_1|\ominus g_P} < (2c + R_{\ominus g_P})/\theta$ and $w_{a_2|\ominus g_P} < (2c + R_{\ominus g_P})/(1-\theta)$. Combining gives $\phi_{g_P}(R_{\ominus g_P}) < 2c + R_{\ominus g_P}$ for all $R_{\ominus g_P}$. Of course, when a_P is replaced with a single link with cost c , the response function is $\phi_{a_P}(R_{\ominus a_P}) = 2c + R_{\ominus a_P}$. \square

Proof of Theorem 7. As always, $\phi_{a_S}(R_{\ominus a_S}) = 2c + R_{\ominus a_S}$. We show that $\phi_{g_S}(R_{\ominus g_S}) \geq \phi_{a_S}(R_{\ominus a_S})$ for fixed $R_{\ominus g_S} = R_{\ominus a_S}$. Because markups are bounded below by 2, $R_{g_P|\ominus g_S} \geq 2(1-\theta_V)c$. We know that $w_{a_V|\ominus g_S}$ is equal to $2\theta_V c + R_{\ominus a_S} + R_{g_P|\ominus g_S}$. Thus, $\phi_{g_S}(R_{\ominus g_S}) = R_{g_P|\ominus g_S} + w_{a_V|\ominus g_S} \geq 2c + R_{\ominus a_S} + R_{g_P|\ominus g_S} \geq \phi_{a_S}(R_{\ominus a_S})$. For the second statement, see that as θ_V decreases, $\phi_{g_P}(\cdot)$ is shifted upwards, while $\phi_{a_V}(\cdot)$ shifts downwards, leading to a decrease in $w_{a_V|\ominus g_S}$, and so in $R_{\ominus g_P|\ominus g_S}$. Similarly, the sensitivity $\phi'_{g_P}(R_{\ominus g_P|\ominus g_S})$ increases (this effect is derived from closed-form expressions of Akgün (2004) for equilibria in an elastic duopoly). For a fixed value of $R_{\ominus g_S}$, $\phi'_{g_S}(R_{\ominus g_S}) = (1 + 3\phi'_{g_P}(R_{\ominus g_P|\ominus g_S})) / (1 - \phi'_{g_P}(R_{\ominus g_P|\ominus g_S}))$, and so is also increasing (the sensitivity is derived using equation (18), presented in the appendix). Integrating $\phi'_{g_S}(R)$ over $[0, R_{\ominus g_S}]$ shows that $\phi_{g_S}(R_{\ominus g_S})$ increase with a decrease in θ_V . \square

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