The Effect of Trading Commissions on Analysts' Forecast Bias*

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Abstract

The paper models the interaction between a sell-side analyst and a risk-averse investor. It studies an analyst's optimal earnings forecast and an investor's optimal trading decision in a setting where the analyst's payoff depends on the trade his forecast generates as well as on his forecast error. The paper shows that in the unique fully separating equilibrium the analyst biases his forecast upward (downward) if his private signal is greater (less) than the firm's stock price.

The model provides several empirical predictions, including: (i) the analyst biases his forecast upward more often than downward and his forecast is on average optimistic; (ii) the analyst acts as if he overweights his private information if it is favorable. If his private information is sufficiently unfavorable, he also acts as if he overweights it, but to a lesser extent. If his private information is slightly unfavorable he acts as if he underweights it; (iii) the analyst's expected squared forecast error may either increase or decrease in the precision of his private information. If the precision of his private information is sufficiently high, a further increase in this precision always increases his expected squared forecast error rather than decreases it (as one might expect).

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1 Introduction

The model explores properties of security analysts' forecast bias and forecast errors if analysts strive to generate revenue from trading commissions. In the model, an analyst forecasts a firm's future earnings based on private information he obtains. He collects trading commissions from an investor that trades upon observing the analyst's forecast. When forecasting earnings, the analyst is not confined to tell the truth but may choose to forecast earnings that differ from his posterior expectations. In the model, forecast errors are costly to the analyst who trades off incentives to generate trading commissions against incentives to provide an accurate estimate of firm's future earnings. This trade-off determines the equilibrium properties of the analyst's forecast errors. Because investors' trade is increasing in the analyst's forecast, unbiased forecasts cannot be sustained in equilibrium for all realizations of the analyst's private signal. This is the case even though investors rationally anticipate the bias in analyst's forecasts – consistent with standard costly signaling models.

In the paper, security prices are set by a continuum of investors with constant absolute risk aversion (CARA). An investor privately obtains a forecast issued by a security analyst and may decide to trade based on this information. We assume that the investor that observes the forecast is a price-taker, i.e. his trade does not affect the stock price. Prior to issuing the forecast, the analyst privately obtains a noisy signal of the firm's future performance. While the model is framed as the analyst issuing an earnings forecast, the analyst may obtain and report on any kind of information about the firm's future performance, as long as the analyst's signal and underlying performance measure are normally distributed. For instance, the performance measure may also refer to price targets, earnings growth or revenue forecasts.

Analysts that issue forecasts face various incentives. In the paper, we focus on sell-side analysts that are employed by full-service brokerage firms and the revenue from trading commissions these analysts may generate for their firms. Typically, analysts' incentives are tied to the brokerage firm's performance through various channels such as bonus payments that depend on the overall performance of the brokerage house, compensation that is determined

¹The model is robust to having multiple investors observe the analyst's forecast, as long as the group of investors is sufficiently small, such that their aggregate demand does not affect the price. Additional modification of the setup in which the informed investors are not price-takers can be based on the setup of Grossman and Stiglitz (1980) or Kyle (1985) (see footnote 11).

by the commissions generated from the trade in the stocks covered by the analyst, equity in the brokerage firm and career concerns. To capture the alignment of the analyst's incentives with the brokerage firm's performance, the model assumes that the analyst's objective function is increasing in his brokerage firm's revenue from trading commissions. In particular, we assume the analyst benefits from a per-share trading commission for the trade his forecast generates.²

The analyst's incentives, however, are not limited to generating trading commissions. As Jackson (2005) states it: "the analyst must trade off the short-term incentive to lie and generate more trade against the long-term gains from building a good reputation." More generally, the analyst also incurs costs from forecast errors. In addition to reputation concerns, these costs might reflect any of the following: the analyst's responsibility to his investors to accurately report on the firm's performance (Morgan and Stocken 2003), increased compensation resulting from higher accuracy (Mikhail et al. 1999), career concerns (Hong and Kubik 2003), or benefits arising from being listed in analyst ranking such as the "All-American Research Team" published by the Institutional Investor magazine (Stickel 1992). In the model, we summarize these incentives by assuming that forecast errors are costly to the analyst and that the cost from forecast errors is any (not necessarily symmetric) U-shaped function that is sufficiently steep at the tails.³

In the analysis, we focus on fully separating equilibria. We show that there exists such equilibrium and that it is unique. In a fully separating equilibrium, the investor can perfectly infer the analyst's bias and private information. In a more realistic setting, one might expect that investors cannot perfectly infer the bias in an analyst's forecast because investors are uncertain about the analyst's "true" objective function in a particular period. We show that the equilibrium is robust to the introduction of additional information asymmetry in the form of a random component to the analyst's cost from forecast errors. Such additional information asymmetry prevents the investor from perfectly inferring the analyst's incentives

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model is motivated as the analyst's payoff being increasing in the trade generated by his forecast,.

However, a broader interpretation of the analyst's payoff function is that his payoff is tied to the change in beliefs induced by his forecast.

³Section 7.8 in the Appendix provides further justification for the analyst's costs from forecast errors. It shows that the analyst's preferred cost function coefficient is not arbitrarily small for the case of a quadratic cost function. It also provides some intuition for why the analyst does not prefer to minimize his cost function coefficient.

and private information (similar to Dye and Sridhar 2004).

In the model's fully revealing equilibrium, the investor does not trade if he incurs positive marginal trading costs and the analyst's private signal is sufficiently close to the firm's stock For such signals, the investor's marginal cost exceeds his marginal benefit from price. trading, resulting in a "no-trading" zone. For all signals that fall within the no-trading zone, the analyst does not bias his forecast. The intuition is that for these signals, the analyst does not generate any trade in a fully separating equilibrium, and hence minimizes his expected cost by choosing zero bias. If the analyst's private signal is to the right of the no-trading zone, he biases his forecast upwards, such that the bias is an increasing, concave and bounded function of his private signal. If the analyst's private signal is to the left of the no-trading zone, he biases his forecast downwards, such that the bias is an increasing, convex and bounded function of his private signal. While the analyst biases his forecast both upward and downward, depending on his private signal, he more often biases his forecast upward than downward. If his cost function is symmetric, the average forecast bias is upward, such that the forecast exceeds the firm's earnings and appears over-optimistic. This prediction is consistent with the majority of empirical studies on analysts' bias (for a review see Kothari 2001).

So far, the model assumed that investor's trades are unrestricted. However, typically short sales are restricted. Introducing a short sale constraint to our model does not affect the analyst's forecasting behavior whenever the short sale constraint is not binding. When the analyst's private signal is sufficiently low, the constraint becomes binding and the analyst's incentive to bias his forecast diminishes. As the analyst's private signal decreases beyond the point where the short sale constraint first binds, the magnitude of the analyst's downward bias decreases and eventually becomes zero.

The fully separating equilibrium of the model gives rise to the following main predictions.⁴ First, an increase in the per share trading commission the analyst receives increases both the magnitude of his bias for any realization of his private signal and the expected forecast bias. As the analyst's per share trading commission converges to zero, the analyst's bias approaches zero as well. Consistent with that prediction, Chen and Jiang (2006) find that

⁴The empirical predictions about the analyst's expected forecast error/bias are derived under the additional assumption that the analyst's cost from forecast error is symmetric around zero.

the deviation of the analyst's forecast from unbiased rational expectations increases when the analyst's benefits from doing so are high and when the costs of doing so are low.

Second, our model predicts that if the analyst's private signal does not fall within a given interval whose upper bound is the prior expectation of the firm's earnings, the analyst issues a forecast as if he overweights his private information, i.e. he places a larger weight on his private information than if he followed Bayes' Rule and truthfully reported his posterior expectation.⁵ If the analyst's private signal falls within the above interval, he issues a forecast as if he underweights his private information. In line with these predictions, Chen and Jiang (2006) provides evidence that analysts who issue forecasts that exceed the consensus overweight their private information. If they issue forecasts that are lower than the consensus, analysts overweight their private information by less and sometimes even underweight it. Easterwood and Nutt (1999) and Friesen and Weller (2006) find that analysts act as if they are overconfident and do not act as if they rationally update their beliefs and truthfully report their expectations.

Third, in the model, an increase in the precision of the analyst's private signal increases the magnitude of the analyst's forecast bias.⁶ The intuition is that a higher precision of the analyst's private signal increases the sensitivity of the investor's demand to the information conveyed in the analyst's forecast. The increased sensitivity of the investor's demand boosts the analyst's incentive to bias his forecast (incentive effect of increased precision).

The final prediction relates the analyst's expected squared forecast error to the precision of his private information. One might expect that analysts that obtain a signal of higher precision issue a forecast that results in a smaller expected squared forecast error (distribution effect). However, the model shows that this does not have to be the case in equilibrium. Instead, the analyst's expected squared forecast error always increases in the precision of the analyst's private information if the precision is sufficiently high. For lower values of the precision of the analyst's signal, an increase in the precision can either increase or decrease the expected squared forecast error, depending on other parameter values of the model. The reason is that in addition to the distribution effect, an increase in the precision of the

⁵The interval is between the midpoint of the no-trading zone and the prior expectation of the firm's earnings. In the model, the midpoint of the no-trading zone turns out to equal the stock price of the firm (under assumption that the firm is liquidated at the end of the period).

⁶This prediction is derived under the additional sufficient assumption that the analyst's marginal cost from forecast error is not concave.

analyst's private signal increases the analyst's incentives to bias his forecast (incentive effect). To summarize, the model predicts that analysts with more precise private information do not necessarily issue forecasts that result in smaller expected squared forecast errors. This might shed some light on the surprising empirical findings that affiliated analysts, who are conjectured to possess more precise information about a firm, do not outperform independent analysts in terms of forecast errors (e.g., Gu and Xue 2007).

In addition to the voluminous body of empirical literature, there are several theoretical papers on analysts' forecasts. Our paper is most closely related to Hayes (1998) and Guttman (2007). While our paper focuses on how analysts' incentives to generate trading commissions affect their forecasting decision, Hayes (1998) studies how incentives to generate trading commissions affect security analysts' decisions in covering stocks and gathering information.⁷ In contrast to Hayes (1998), we assume that the precision of the analyst's information is exogenously given, but we allow the analyst to intentionally bias his forecast. That is, the analyst may issue a forecast that deviates from his private information. Both Hayes (1998) and our model assume that the timing of analysts' forecasts is exogenous. Guttman (2007) endogenizes the timing of analysts' forecast in the presence of competition between analysts. In a setup similar to the one in this paper, Guttman (2007) finds that analysts with more precise initial private information tend to issue their forecasts earlier. Ideally, a model would allow an analyst to choose the accuracy of his private information (information gathering decision), the bias in his forecast (reporting decision) and when to release his private information (timing decision).

The paper proceeds as follows. Section 2 introduces the model. Section 3 derives the equilibrium of the model. In this section, we start with the simplifying assumption that investors do not incur any marginal trading costs. Next, we relax this assumption and derive the equilibrium for positive marginal trading costs. Then, we introduce short-sale constraints and finally we analyze a case where there is additional information asymmetry with respect to the analyst's payoff function. We show that the equilibrium is robust to these extensions. Section 4 assumes a quadratic cost function which enables us to derive a closed form solution to analyst's equilibrium reporting strategy. Section 5 presents empirical predictions. Section

⁷Irvine (2004) tests empirical predictions based on Hayes (1998) and also examines how investors' trading decisions relate to the absolute value of analysts' ex-post forecast errors.

6 concludes. The Appendix contains formal proofs and additional analysis.

2 Setup

This section describes a parsimonious model of an analyst who issues an earnings forecast and a risk-averse investor who trades based on the analyst's forecast. The sequence of events is the following. A firm generates earnings, x, that distributed normally with mean μ_x and precision τ_x (i.e. $\tau_x = \frac{1}{Var(x)}$). Let f(x) denote the probability density function of the firm's earnings and assume it is common knowledge. The realized value of the firm's earnings are not directly observable to anybody outside the firm. However, an analyst obtains private information about the firm's earnings. His private information, ψ , is a noisy signal of firm's earnings. In particular, $\psi = x + \varepsilon$ where ε is independently and normally distributed with zero mean and precision τ_{ε} . At times we refer to the realized signal ψ as the analyst's type. Based on his private information, the analyst provides a forecast, x^{R} , of the firm's future earnings to an investor. When issuing his forecast, the analyst is not confined to tell the truth. Rather, he can release a forecast that differs from his own beliefs about the firm's expected earnings given his private information ψ . incentive to issue a forecast that differs from his personal beliefs arises due to the trading commissions he collects if the investor trades based on the information conveyed by the This provides an incentive to issue a forecast that generates as much trade as forecast. possible. However, the analyst incurs a personal cost whenever his forecast differs from the realization of the firm's earnings, i.e., forecast errors are costly to the analyst. When deciding what forecast to issue, the analyst trades-off the expected cost from forecast errors against trading commissions following a given forecast.

In particular, the analyst's cost from forecast errors is $g\left(x^R-x\right)$ where $g\left(\cdot\right)$ is a twice-differentiable, convex function where for any given $x\lim_{x^R\to-\infty}\frac{\partial g\left(x^R-x\right)}{\partial x^R}=-\infty$ and $\lim_{x^R\to\infty}\frac{\partial g\left(x^R-x\right)}{\partial x^R}=\infty$. These conditions imply that the cost function is a "U-shaped" function with a unique minimum and that for a given sign of the forecast error, the marginal cost increases in the magnitude of the forecast error. We assume that the forecast that minimizes the analyst's expected cost is the unbiased forecast, i.e., issuing a forecast that equals the analyst's ex-

⁸Note that the results remain unchanged if the limit of $\frac{\partial g(x^R-x)}{\partial x^R}$ for $x^R \to +/-\infty$ is bounded by a finite constant that depends on the other parameters of the model.

pectation given his private information. However, a similar equilibrium exists if we relax this assumption and allow for a different cost minimizing bias.

The analyst's trading commission is proportional to the trading volume generated by his forecast. The trading volume is the difference between the investor's initial demand, D_0 , and his demand after he observes the analyst's forecast, $D_1(x^R)$. We denote the trading commission that the analyst obtains per traded share as c_A . To summarize, when issuing his forecast, x^R , given his private information, ψ , the analyst maximizes the following objective function

$$u^{A}\left(x^{R},\psi\right) = c_{A}\left|D_{1}\left(x^{R}\right) - D_{0}\right| - E\left[g\left(x^{R} - \widetilde{x}\right)\right]\psi\right]. \tag{1}$$

Apart from the investor that receives the analyst's forecast ("informed investor"), there is a continuum of uninformed investors, each of which has a utility function with constant absolute risk aversion given by $-e^{-\rho W_1}$ where W_1 is the end-of-period wealth and ρ is the risk-aversion coefficient. It is a well-known result that given the above assumptions the initial demand of the representative investor is $D = \frac{\mu_x - P_0}{\rho} \tau_x$ where P_0 is the equilibrium stock price. The exogenous per capita supply of the firm's stock is given by S. In equilibrium the firm's stock price, P_0 , is set such that the per capita demand, D, equals the per capita supply S. The equilibrium stock price, that equates demand and supply, is $P_0 = \mu_x - \frac{\rho S}{\tau_x}$.

We assume that the informed investor is sufficiently small such that he acts as a price taker and can buy or sell any number of shares at the stock price P_0 . The number of shares the investor buys or sells, i.e., the trading volume, is given by $|D_1 - D_0|$ where D_0 is the informed investor's initial holding. For each share that the investor buys or sells following the analyst's forecast, he incurs per-share trading costs of c_I . After observing the analyst's forecast x^R , the informed investor updates his beliefs about the firm's earnings in a rational, Bayesian manner and chooses the demand that maximizes his expected utility which is given by

$$D_{1}(x_{R}) \in \underset{D_{1}}{\operatorname{arg\,max}} E\left[u^{I}(D_{1}) | x^{R}\right] = -\int_{-\infty}^{\infty} e^{-\rho(W_{o} + D_{1}(\widetilde{x} - P_{0}) - c_{I}|D_{1} - D_{0}|)} f\left(\widetilde{x} | x^{R}\right) dx. \quad (2)$$

⁹We assume that the investor is allowed to trade only once (similar to Grossman and Stiglitz 1980). Alternatively, we could assume that either the firm's earnings themselves or some other information about the firm's earnings are revealed immediately after the investor trades upon the analyst's forecast. This ensures that the investor is exposed to some risk and hence will not trade an infinite amount.

¹⁰Depending on the analyst's incentive system, c_A may or may not equal c_I . While we allow for $c_I \neq c_A$, we assume that both c_I and c_A are the same for buys and sells. However, this is only a simplifying assumption and introducing different marginal costs for sells and buys will result in a similar equilibrium.

Anticipating his holdings $D_1(x^R)$ following the analyst's forecast, the investor chooses his optimal initial holdings D_0 that maximizes his expected utility.¹¹

All parameters of the model are assumed to be common knowledge.

3 Equilibrium

We can now define the Perfect Bayesian Equilibrium of this game.

Definition 1 The equilibrium consists of the analyst's forecasting strategy $x^R(\psi) : \mathbb{R} \to \mathbb{R}$ and the informed investor's demand function $D_1(x^R) : \mathbb{R} \to \mathbb{R}$ and D_0 such that

- (i) For all ψ , the analyst's forecasting strategy $x^R(\psi)$ maximizes his expected utility in (1) where the analyst's correctly anticipates the informed investor's equilibrium demand, $D_1(x^R)$.
- (ii) For any x^R, the informed investor's demand D₁ (x^R) maximizes his expected utility in
 (2) where the informed investor's beliefs about the firm's earnings are consistent with
 x^R (ψ) using Bayes rule, whenever applicable.
- (iii) Taking his demand $D_1(\cdot)$ and the analyst's forecasting strategy $x^R(\cdot)$ as given, the informed investor chooses D_0 that maximizes his ex-ante expected utility.

We focus on equilibria where the analyst's forecast fully reveals his private signal (fully separating equilibria). In such an equilibrium, the optimal initial demand of the informed investor D_0 is the same as the demand of the representative uninformed investor, D^{12} As a result, whether the informed investor anticipates that he will obtain the analyst's forecast has no effect on the informed investor's initial holding and on the equilibrium as a whole.

 $^{^{-11}}$ A natural modification of the setup in which the informed investors are not price-takers could be based on the setup of Grossman and Stiglitz 1980 (herafter G&S). In such a setup, the analyst sells his forecast to a substantial fraction of investors who pay a fixed fee for obtaining the analyst's forecast in addition to the trading commission per share. For $c_I = 0$, the investors' demand is exactly the same as in G&S (see section 3.1) and one can show that there exists an equilibrium in which the analyst's forecasting strategy remains qualitatively unchanged. For $c_I > 0$, the informed investors' demand is no longer linear in the inferred private signal due to the no-trading zone (see section 3.2). As a result, the original linear equilibrium of G&S no longer holds. However, as long as there exists an equilibrium for the G&S setup with non-linear demand where the informed investors' trade is increasing in the private information, the analyst's optimization problem remains qualitatively the same. Similar arguments apply to the setup of Kyle (1985).

¹²We include a proof of this result in the Appendix.

3.1 Zero marginal trading cost

Further, we initially assume for simplicity that $c_I = 0$, i.e., the investor does not pay any per share trading commission (or ignores this commission while determining his demand). We later analyze the case of $c_I > 0$ and show that the major characteristics of the equilibrium remain qualitatively unchanged.

In a fully separating equilibrium the firm's earnings given the analyst's forecast are normally distributed and the informed investor's demand following a forecast x^R is

$$D_1(x^R) = \frac{E\left[\widetilde{x}|x^R\right] - P_0}{\rho Var\left(\widetilde{x}|x^R\right)}.$$

In such equilibria, there is a signal ψ^* following which the investor's demand remains unchanged. The analyst that observes the signal ψ^* does not generate any trade in the fully separating equilibrium, and hence issues the forecast that minimizes his expected cost from forecast errors, i.e., he issues an unbiased forecast.

Before we show that a unique fully separating equilibrium exists, we derive several properties that the analyst's forecasting strategy must possess to be part of an equilibrium. First, we show that the analyst's forecasting strategy $x^R(\psi)$ has to be continuous in his private information, and hence, the analyst's equilibrium forecast must be strictly increasing in his private signal. Next, we show that not only the analyst's forecast, x^R , but also his forecast bias, $b(\psi) = x^R(\psi) - E[\tilde{x}|\psi]$, is increasing in the analyst's signal. Nevertheless, the bias is bounded from both below and above. Based on these characteristics, we show that there exists a fully separating equilibrium and that it is unique.

Below we provide a more detailed argument of the steps outlined above. We start with identifying the signal ψ^* following which no trade occurs. In a fully separating equilibrium, the analyst's forecast, $x^R(\psi)$, for any ψ is informationally equivalent to his private information. Hence, in equilibrium the informed investor's demand is given by $D_1(x^R) = D_1(\psi) = \frac{E[\widetilde{x}]\psi] - P_0}{\rho Var(\widetilde{x}|\psi)}$ where $E[\widetilde{x}]\psi] = \frac{\mu_x \tau_x + \psi \tau_\varepsilon}{\tau_x + \tau_\varepsilon}$ and $\frac{1}{Var(\widetilde{x}|\psi)} = \tau_x + \tau_\varepsilon$. The signal ψ^* is the signal for which $D_1(\psi^*) = D_0$ and no trade occurs. It turns out that ψ^* is independent of the precision of the analyst's private signal and is given by

$$\psi^* = P_0. \tag{3}$$

That is, regardless of the precision of the analyst's signal, the informed investor will not trade if he infers a private signal of $\psi^* = P_0$. We denote investor's expectation if he infers

that $\psi = \psi^*$ by x^* where $x^* = E\left[\tilde{x}|\psi^*\right]$. Note that x^* , the posterior expectation for which the investor does not trade, is lower than μ_x . If the informed investor infers a signal $\psi > \psi^*$ ($\psi < \psi^*$), he will buy (sell) additional shares and the number of shares traded will be increasing (decreasing) in the inferred signal ψ .

Claim 1 In a fully separating equilibrium, following a signal ψ^* the analyst will not bias his forecast, i.e., $x^R(\psi^*) = E[\widetilde{x}|\psi^*] = x^*$.

In a fully separating equilibrium the private signal is correctly inferred by the investors, hence, if $\psi = \psi^*$ no trade will occur. From the analyst's perspective, ψ^* is the signal that generates the lowest payoff ("lowest type"). Hence, in equilibrium the analyst will not be willing to bear any "signaling costs" in the form of forecast bias.

Next, we show that the forecast is continuously increasing in the analyst's private signal ψ .

Claim 2 The analyst's forecast x^R is continuously increasing in ψ .

In a fully revealing equilibrium, the analyst's trading commission is continuous in ψ because the investor can perfectly infer the analyst's private signal. In equilibrium, it cannot be the case that there is a discrete jump in the bias at any private signal ψ' . If there were a jump at ψ' the difference between the expected cost from forecast errors for types just to the right of ψ' and just to the left of ψ' is discrete, while the difference between the trade generated by the above two types is arbitrarily small. Therefore, the type with the higher expected cost from forecast errors would always have an incentive to mimic the other type, which contradicts the existence of an equilibrium with discontinuous bias.

Given that the forecast is continuous in the analyst's private information, in a fully revealing equilibrium the analyst's forecast must be strictly monotone in ψ . The unbounded support of the analyst's private signal in combination with unbounded cost from forecast errors preclude that in equilibrium the analyst forecast is monotonically decreasing in ψ .

In the next two steps, we establish that not only the analyst's forecast but also the bias in his forecast is strictly increasing in ψ .

Claim 3 For any $\psi > \psi^*$ ($\psi < \psi^*$), the bias $b(\psi)$ is greater (less) than $b(\psi^*)$.

Since the analyst's forecast is increasing in ψ , for any $x^R > x^R (\psi^*)$ the informed investor infers that $\psi > \psi^*$ and his trade is increasing in the analyst's forecast. Hence, by increasing the bias in his forecast, the analyst obtains higher trading commissions. In order for this to be sustainable in equilibrium, the analyst's expected cost from biasing his forecast must also be increasing in the forecast bias. Since the cost minimizing bias equals zero for all ψ , expected costs are increasing in the bias only if the bias is positive.¹³ So, in equilibrium, the bias is always upwards (positive) if $\psi > \psi^*$. For $\psi < \psi^*$ a similar argument implies that the bias is always downwards (negative).

Claim 4 The bias is weakly increasing in ψ .

From Claim 2 and 3 it follows that the bias is continuously increasing in a neighborhood around ψ^* . When the bias increases in ψ , $E\left[\widetilde{x}|x^R\right]$ increases in the forecast x^R at rate of less than 1 (but positive).¹⁴ Suppose the bias starts to decrease in ψ at some point ψ' . When the bias decreases, the conditional expectation increases in the report at a rate of more than 1. If the bias starts to decrease at $\psi' > \psi^*$ then there are two types, one to the left and one to the right of ψ' , that have the same bias and hence the same marginal expected cost from forecast errors. However, the marginal benefit of the two types differs because $E\left[\widetilde{x}|x^R\right]$ (and hence the trading volume) increases at a lower rate for the type to the left of ψ' than for the type to the right of ψ' . This contradicts the equilibrium condition that marginal cost equals marginal benefit for any type. Based on a similar argument, we preclude a decreasing bias for $\psi < \psi^*$. Hence, the bias is increasing in ψ for all ψ .

Claim 5 The bias function, $b(\psi)$, is concave in ψ for $\psi > \psi^*$ and convex in ψ for $\psi < \psi^*$.

We know from Claim 4 that the bias is increasing in ψ . Hence, for $\psi > \psi^*$ both the expected cost and the expected marginal cost from forecast errors are increasing in ψ .

¹³For a given bias, the expected cost from forecast errors is independent of the analyst's private information ψ . To see that, we can rewrite $g\left(x^R-x\right)$ as $g\left(b+v\right)$ where $b=x^R-E\left[\widetilde{x}|\psi\right]$ and $v=E\left[\widetilde{x}|\psi\right]-x$. It follows that v given any realized private signal, ψ , is distributed normally with mean zero and precision $\tau_x+\tau_\varepsilon$. The expected cost from forecast error, $\int_{-\infty}^{\infty}g\left(x^R-x\right)f\left(x|\psi\right)dx$, can be rewritten as $\int_{-\infty}^{\infty}g\left(b+v\right)f\left(v\right)dv$. Hence, the expected cost for a given bias is independent of the realized signal ψ .

¹⁴To see this consider that $E\left[\widetilde{x}|x^R\right] = x^R - b\left(x^{R-1}\left(x^R\right)\right)$ where $x^{R-1}\left(\cdot\right)$ denotes the inverse function of the analyst's forecasting strategy $x^R\left(\psi\right)$. Hence, $\frac{dE\left[\widetilde{x}|x^R\right]}{dx^R} = 1 - \frac{db}{d\psi}\frac{dx^{R-1}}{dx^R} = 1 - \frac{db}{d\psi}\frac{1}{dx^R\left(\psi\right)/d\psi}$. The claim follows immediately from $\frac{\partial x^R\left(\psi\right)}{\partial \psi} > 0$.

Suppose, the bias was convex in ψ for $\psi > \psi^*$. Then, the sensitivity of informed investor's expectation about the firm's earnings, $E\left[\widetilde{x}|x^R\right]$, to changes in the analyst's forecast would be lower for higher values of x^R (i.e. $\partial E\left[\widetilde{x}|x^R\right]/\partial x^R$ would be weakly decreasing in x^R). This would imply that the marginal benefit from trading commissions was lower for greater values of ψ (and $x^R(\psi)$). Since in equilibrium the marginal expected cost from biasing the forecast must equal the marginal benefit from trading commission for any $x^R(\psi)$, the above opposite effects under the assumption of a convex bias function cannot be sustained in equilibrium for $\psi > \psi^*$. Again, we can make a similar argument for the bias being convex for $\psi < \psi^*$.

Claim 6 The bias is bounded from below and above.

Since the bias increases in ψ , the informed investor's expectations about firm's earnings, $E\left[\widetilde{x}|x^R\right]$, increases in the forecast at a rate of less than one. Hence, the marginal benefit from biasing the forecast is bounded, which implies that the marginal expected cost (which equals the marginal benefit in equilibrium) is bounded as well. Since the cost function, $g\left(\cdot\right)$, is sufficiently steep in its tails, this implies that the bias itself is also bounded.

Using the equilibrium properties of the analyst's forecasting strategy that we have established so far, we can now state the Proposition that establishes both existence and uniqueness of a fully separating equilibrium.

Proposition 1 There exists a unique fully separating equilibrium where

- (i) The analyst's equilibrium forecasting strategy is $x^R(\psi) = E[\tilde{x}|\psi] + b(\psi)$ where the bias function $b(\psi)$ is increasing, continuous, convex for $\psi < \psi^*$, concave for $\psi > \psi^*$, bounded from above and below and $b(\psi^*) = 0$;
- (ii) The investor's demand following the analyst's forecast is given by

$$D_1(x^R) = \frac{x^R - b(x^R) - P_0}{\rho Var(x|\psi)}$$

where $b\left(x^{R}\left(\psi\right)\right)=b\left(\psi\right)$ is the investor's beliefs about the analyst's bias given x^{R} , stock price $P_{0}=\mu_{x}-\frac{\rho S}{\tau_{x}}$.

(iii) The investor's initial holding is $D_0 = D = S$.

Figures 1 and 2 illustrate the equilibrium in Proposition 1. To generate the figures we used the quadratic cost function analyzed in section 4 with parameter values $c_A = \rho = \tau_x = \tau_{\varepsilon} = S = 1$; $\mu_x = 2$ (which implies $\psi^* = 1$). The figure presents the analyst's forecast bias, $b(\psi)$, and his forecasting strategy, $x^R(\psi)$.

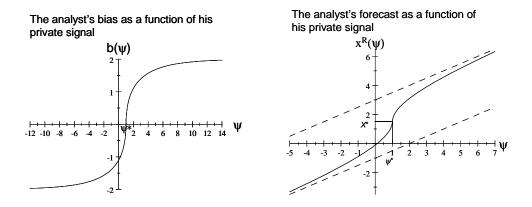


Figure 1: The equilibrium bias and forecasting strategy of the analyst.

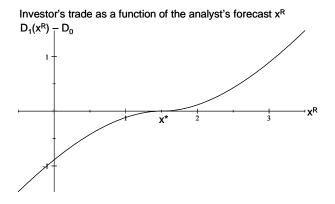


Figure 2: The equilibrium trade as a function of the forecast.

The only part of the proposition that hasn't been established in the preceding claims is that such fully separating equilibrium exists and is unique. To see the uniqueness, suppose there existed two equilibria. With out loss of generality assume that the two equilibria differ for some $\psi > \psi^*$. For both equilibria the boundary condition that the bias at ψ^* equals zero has to hold. Since both equilibria start out with the same boundary condition, there has to be an interval of types for which in one equilibrium the bias is greater and increases at a higher rate relative to the other. A greater bias implies that the marginal expected cost,

 $\frac{\partial E[g(E[x|\psi]+b-x)|\psi]}{\partial b}$, is higher. Similarly, the bias being increasing at a higher rate implies that the marginal benefit from trading commission, $\frac{\partial}{\partial b} \frac{c_A E\left[\tilde{x}|x^R\right]}{\rho Var(\tilde{x}|x^R)}$, is lower. The intuition for the latter is that if the bias increases at a higher rate, the informed investor attributes a larger part of the increase in the forecast to the increase in the bias rather than to the increase in expected firm value. Hence, if one of the bias functions constitutes an equilibrium then the other bias function cannot be part of an equilibrium. This is equivalent to there being a unique equilibrium. We leave the proof of existence to the Appendix.

3.2 Positive marginal trading cost

So far, we assumed that the informed investor does not incur costs from trading. A more realistic assumption is that he incurs positive marginal trading costs, for instance trading commissions that he has to pay to the analyst's brokerage house. To capture this assumption we allow for $c_I > 0$ in the following.

Proposition 2 There exists a unique fully separating equilibrium, in which the analyst's forecasting strategy is given by

$$b_{c_{I}}(\psi) = \begin{cases} b_{0}\left(\psi - c_{I}\frac{\tau_{x} + \tau_{\varepsilon}}{\tau_{\varepsilon}}\right) & \text{if } \psi > P_{0} + c_{I}\frac{\tau_{x} + \tau_{\varepsilon}}{\tau_{\varepsilon}} \\ 0 & \text{if } P_{0} + c_{I}\frac{\tau_{x} + \tau_{\varepsilon}}{\tau_{\varepsilon}} > \psi > P_{0} - c_{I}\frac{\tau_{x} + \tau_{\varepsilon}}{\tau_{\varepsilon}} \end{cases}$$

$$b_{0}\left(\psi + c_{I}\frac{\tau_{x} + \tau_{\varepsilon}}{\tau_{\varepsilon}}\right) & \text{if } P_{0} - c_{I}\frac{\tau_{x} + \tau_{\varepsilon}}{\tau_{\varepsilon}} > \psi$$

where $b_0(\psi)$ defines the analyst's forecasting strategy for $c_I = 0$. The informed investor's demand function is given by

$$D_{1}\left(x^{R}\right) = \begin{cases} \frac{E\left[\tilde{x}|x^{R}\right] - P_{0} - c_{I}}{\rho/(\tau_{x} + \tau_{\varepsilon})} & \text{if } x^{R} > \frac{\mu_{x}\tau_{x} + P_{0}\tau_{\varepsilon}}{\tau_{x} + \tau_{\varepsilon}} + c_{I} \\ D_{0} & \text{if } \frac{\mu_{x}\tau_{x} + P_{0}\tau_{\varepsilon}}{\tau_{x} + \tau_{\varepsilon}} + c_{I} > x^{R} > \frac{\mu_{x}\tau_{x} + P_{0}\tau_{\varepsilon}}{\tau_{x} + \tau_{\varepsilon}} - c_{I} \end{cases}$$

$$\frac{E\left[\tilde{x}|x^{R}\right] - P_{0} + c_{I}}{\rho/(\tau_{x} + \tau_{\varepsilon})} & \text{if } \frac{\mu_{x}\tau_{x} + P_{0}\tau_{\varepsilon}}{\tau_{x} + \tau_{\varepsilon}} - c_{I} > x^{R}$$

$$\text{where } P_{0} = \mu_{x} - \frac{\rho S}{\tau_{x}}, \ D_{0} = D \ \text{and } E\left[\tilde{x}|x^{R}\right] = x^{R} - b_{c_{I}}\left(x^{R}\right). \quad \text{The inferred bias } b_{c_{I}}\left(x^{R}\right) \text{ is }$$

defined by

$$b_{c_I}\left(x^R\right) = \begin{cases} b_0\left(x^R - c_I\right) & \text{if } x^R > \frac{\mu_x \tau_x + P_0 \tau_\varepsilon}{\tau_x + \tau_\varepsilon} + c_I \\ \\ 0 & \text{if } \frac{\mu_x \tau_x + P_0 \tau_\varepsilon}{\tau_x + \tau_\varepsilon} + c_I > x^R > \frac{\mu_x \tau_x + P_0 \tau_\varepsilon}{\tau_x + \tau_\varepsilon} - c_I \\ \\ b_0\left(x^R + c_I\right) & \text{if } \frac{\mu_x \tau_x + P_0 \tau_\varepsilon}{\tau_x + \tau_\varepsilon} - c_I > x^R \end{cases}$$

Following the introduction of marginal trading costs to the investor, we obtain a "no-trading" zone. That is, if the investor infers that the analyst observed a signal close to ψ^* the investor's marginal benefit from trading is lower than his marginal cost, c_I . Hence, in equilibrium, he will not trade at all. In the fully separating equilibrium, the analyst anticipates that he will not generate trading commission for those realizations of his private signal. As a result, he will not bear any cost for biasing his forecast. This is reflected in his equilibrium forecasting strategy which shows that he will truthfully report his private beliefs for all $\psi \in \left[\psi^* + c_I \frac{\tau_x + \tau_\varepsilon}{\tau_\varepsilon}, \psi^* - c_I \frac{\tau_x + \tau_\varepsilon}{\tau_\varepsilon}\right]$. For notational convenience let Ψ_{NT} denote the interval $\left[\psi^* + c_I \frac{\tau_x + \tau_\varepsilon}{\tau_\varepsilon}, \psi^* - c_I \frac{\tau_x + \tau_\varepsilon}{\tau_\varepsilon}\right]$ for which no trade occurs – the no-trading zone.

For all realizations outside of the no-trading zone, the equilibrium that we obtain for $c_I > 0$ is equivalent to the equilibrium that we obtained in Proposition 1 (when the investor did not incur any costs from trading). The analyst's bias function and the equilibrium trading are identical in the two equilibria when the bias and the demand are expressed as a function of the distance between the private signal and boundary of the no-trading zones (where in the equilibrium for $c_I = 0$ the no trading zone is the singleton ψ^*).

In particular, if $c_I > 0$ the analyst's bias and the investor's demand that occur in equilibrium for a signal that exceeds the upper bound of the no-trading zone by a certain amount are exactly the same as the analyst's bias and the investor's demand that occur in equilibrium if $c_I = 0$ for a signal that exceeds the "no-trading" point, ψ^* , by the same amount. Figure 3 illustrates that.

It might seem surprising that the functional form of the equilibrium bias and demand remains unchanged. When the investor does not incur any costs from trading, he optimally buys additional shares if his posterior expectations about firm's earnings, $E\left[\tilde{x}|x^R\right]$, exceeded x^* (the posterior beliefs for which no trade occurs even if $c_I = 0$). Similarly, he optimally chose to sell shares if his posterior expectations $E\left[\tilde{x}|x^R\right] < x^*$. When the investor does incur costs from trading, $c_I > 0$, he buys additional shares only if his posterior expectation is sufficiently high to compensate him for the trading costs of c_I . That is, he only buys shares if his posterior expectations, $E\left[\tilde{x}|x^R\right]$, exceed $x^* + c_I$. Correspondingly, he only sells shares if his posterior expectation is sufficiently low to justify the trading costs of c_I . That is, he only sells shares if $E\left[\tilde{x}|x^R\right]$ is less than $x^* - c_I$. If his posterior expectation fall inbetween these two cutoffs (i.e. fall within the no-trading zone), the investor will not

trade (see equation 4). The introduction of positive trading costs per share does not affect the sensitivity of the investor's demand function with respect to the private signal he infers from the analyst's forecast. In particular, for $E\left[\widetilde{x}|x^R\right]$ outside of the no trading zone the investor's demand is still a linear function of the inferred private signal with the same slope as in the case where there were no trading costs. Figure 3 compares the demand for the cases of zero and positive trading costs.

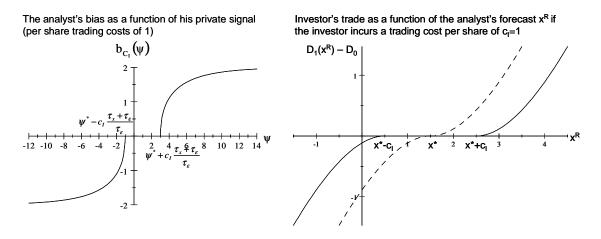


Figure 3: Positive Trading Costs

For all signals for which the posterior expectations fall into the no trading zone, the analyst's forecast will not generate any trade in a fully separating equilibrium, and hence he will choose not to bias his forecast. The intuition is exactly the same as the intuition provided for claim 1.

Since the equilibrium demand to the right (left) of the no trading zone for $c_I > 0$ is the same as the demand to the right (left) of the "no trading point" x^* for $c_I = 0$, the analyst achieves the same change in demand by changing the investor's expectations by a certain amount. As a result, in equilibrium, the analyst faces the same incentives from trading commissions and the same costs from biasing his forecast if the investor incurs positive trading costs as if the investor does not incur any trading costs.

3.3 Short-sale constraints

So far, we assumed that the investors' ability to sell and buy shares is symmetric and unconstrained. However, typically financial institutions essentially limit the number of

shares investors may short-sell (or it is too expensive to do so). If the number of shares the investor can sell after he obtains the analyst forecast is limited, the commission the analyst can generate from issuing a low forecast is restricted as well. Hence, a short-sale constraint may affect the analyst's incentive to bias his forecast if he obtains a low signal. In the following we characterize the effect of a short-sale constraint on the equilibrium.

Suppose the investor can infer the analyst's private signal in equilibrium. Then, the short sale constraint becomes binding for ψ sufficiently low. In particular, let $\underline{\psi}$ denote the highest realization of the analyst's signal for which the short sale constraint is binding. The equilibrium we derive is such that for all $\psi > \underline{\psi}$ the analyst's forecast fully reveals his private information. In addition, for all $\psi > \underline{\psi}$ the analyst's equilibrium bias and the trade it generates are the same as in the absence of any short sale restrictions. For all private signals $\psi \leq \underline{\psi}$ the analyst chooses the forecast that minimizes his expected cost from forecast error subject to the constraint that the forecast induces investors to short-sell the maximum amount of shares. This implies that the analyst reports $x^R(\underline{\psi})$ for all ψ such that the conditional expectation, $E[\widetilde{x}|\psi]$, exceeds $x^R(\underline{\psi})$. For all lower values of ψ the analyst reports an unbiased forecast. This forecasting strategy ensures the analyst the maximum trading commission for all $\psi \leq \underline{\psi}$. Figures 4 and 5 illustrate the analyst's equilibrium forecast bias and investor's demand in the presence of a short-sale constraint of s shares. Here, s is the maximum number of shares an investor can short-sell. As a result, the maximum number of shares he can sell is the sum of s and his initial holdings D_0 .

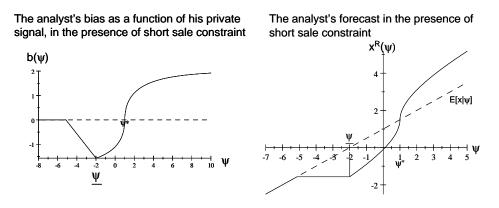


Figure 4: Analyst's bias and forecast in the presence of short sale constraint

The equilibrium we describe above includes a pooling interval to the left of $\underline{\psi}$ (in the size

Investor's trade as a function of the analyst's forecast x^R, in the presence of short sale constraint

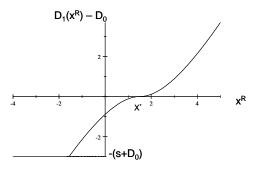


Figure 5: Investor's demand in the presence of short sale

of $-\frac{\tau_x+\tau_\varepsilon}{\tau_\varepsilon}b\left(\underline{\psi}\right)$). We cannot preclude the existence of other equilibria that include different pooling intervals. However, the equilibrium that we derive is the unique equilibrium in which the informed investor's equilibrium demand would be the same if he were to observe the analyst's private information. In that sense, the equilibrium we derive dominates other potential equilibria from the investor's perspective.¹⁵

3.4 Unknown analyst's objective function

So far, the analyst's payoff function was common knowledge and the information asymmetry between the analyst and the investor was limited to the analyst's private signal ψ . This resulted in an equilibrium in which the fully separating strategy of the analyst fully revealed his private signal. In a more realistic setting, one might expect that investors' are uncertain about the analyst's "true" objective function in a particular period. To verify the robustness of the model to this additional information asymmetry, we introduce an additional parameter to the analyst's payoff function, about which the investor is uninformed (similar to Dye and Sridhar 2004). In particular, we assume that the analyst's payoff function is

$$u^{A}\left(x^{R},\psi,\upsilon\right)=c_{A}\left|D_{1}\left(x^{R}\right)-D_{0}\right|-E\left[g\left(x^{R}-\widetilde{x}-\upsilon\right)\right|\psi,\upsilon\right]$$

where v is normally distributed with mean zero and precision τ_v , and is independent of x and ψ . While the analyst observes v before deciding about his forecast, the investor does

¹⁵We cannot preclude that the analyst incurs lower expected cost from forecast errors in another partially pooling equilibrium, if such exists. Hence, if such other partially pooling equilibrium exist we cannot rank the equilibria in terms of Pareto Dominance.

not observe v. In the Appendix, we show that the equilibrium in Proposition 1 is robust to the additional information asymmetry about the analyst's payoff function. In particular, we show the existence of an equilibrium where the investor is no longer able to perfectly infer the analyst's private signal ψ about the firm's earnings. Instead, the investor can infer only the sum of the analyst's posterior expectation and v, that is the investor learns $E\left[\widetilde{x}|\psi\right] + v$. For more details and formal proof see the appendix.

4 Equilibrium with a quadratic cost function

So far, the analyst's cost function, $g(\cdot)$, was fairly general in its form. While we were able to derive several characteristics of the analyst's forecast bias in the fully separating equilibrium, it is impossible to derive a closed form solution for the equilibrium forecast and demand. In this section, we trade off the general form of the cost function for a more specific cost function that enables us to derive a closed form solutions to the analyst's and investor's equilibrium maximizations problems. The closed form solution enables us to generate a specific example of the equilibrium forecasting and trading behavior which we use as a basis for all the plots.

In particular, we assume that the analyst's cost function is quadratic in his forecast error, a fairly common assumption in the disclosure literature. So, the analyst's payoff function is

$$u^{A}\left(x^{R},\psi\right) = c_{A}\left|D_{1}\left(x^{R}\right) - D_{0}\right| - \frac{1}{2}E\left[\left(x^{R} - \widetilde{x}\right)^{2}\middle|\psi\right]. \tag{5}$$

For simplicity of disposition, we assume that the informed investor does not face any short-sale constraint and that trading is costless $(c_I = 0)$. As in the case with general cost functions, the equilibrium can easily be extended to account for short-sale constraint and positive trading costs. Since the quadratic cost function is a particular case of the more general convex cost function $g(\cdot)$, all equilibrium characteristics established so far hold for the quadratic cost function as well. In particular, the equilibrium bias $b(\psi)$ is increasing, continuous, convex for $\psi < \psi^*$, concave for $\psi > \psi^*$, bounded from above and below, and zero for ψ^* .

As in the general case, the analyst does not bias his forecast if he obtains ψ^* as his private signal. Using $b(\psi^*) = 0$ as a boundary condition for the differential equation for $b(\psi)$ (that stems from the first order condition of the analyst's optimization problem), we can obtain a

closed form solution for the analyst's forecasting strategy. Recall that the demand $D_1\left(x^R\right)$ equals $\frac{x^R - \hat{b}\left(x^R\right) - P_0}{\rho Var(\tilde{x}|\psi)}$ where $\hat{b}\left(x^R\right)$ denotes the bias the investor infers if he observes a forecast x^R . Then, the first order condition with respect to x^R for $\psi > \psi^*$ is x^{16}

$$c_{A} \frac{1 - \frac{\partial \widehat{b}(x^{R})}{\partial x^{R}}}{\rho Var\left(\widetilde{x}|\psi\right)} - E\left[x^{R} - \widetilde{x}|\psi\right] = 0$$

In a fully separating equilibrium, the investor's beliefs are consistent with the analyst's forecasting strategy, that is $\hat{b}(x^R(\psi)) = x^R(\psi) - E[\tilde{x}|\psi] \equiv b(\psi)$. Substituting into the first order condition yields

$$c_{A} \frac{1 - b'(\psi) \frac{1}{\frac{\partial E\left[\widetilde{x}|\psi\right]}{\partial \psi} + b'(\psi)}}{\rho Var\left(\widetilde{x}|\psi\right)} - b(\psi) = 0$$

Given the boundary condition $b(\psi^*) = 0$, the solution to the above differential equation yields the equilibrium forecast bias function for $\psi > \psi^*$. Since the quadratic cost function is symmetric around ψ^* solving for the equilibrium forecast bias for $\psi < \psi^*$ yields a similar differential equation and the absolute value of the equilibrium bias is symmetric relative to ψ^* . The following Proposition describes the equilibrium for the case of the quadratic cost function.

Proposition 3 There exists a unique fully separating equilibrium where

(i) The analyst's optimal forecast is given by $x^{R}(\psi) = E\left[\widetilde{x}|\psi\right] + b(\psi)$ where t^{17}

$$b\left(\psi\right) = \begin{cases} \frac{c_{A}(\tau_{x} + \tau_{\varepsilon})}{\rho} \left(1 + \operatorname{Pr} oductLog\left(-e^{-\frac{\left(\psi - \psi^{*}\right)\tau_{\varepsilon}\rho}{c_{A}(\tau_{x} + \tau_{\varepsilon})^{2}} - 1}\right)\right) & \text{if } \psi \geq \psi^{*} \\ -\frac{c_{A}(\tau_{x} + \tau_{\varepsilon})}{\rho} \left(1 + \operatorname{Pr} oductLog\left(-e^{-\frac{\left(\psi^{*} - \psi\right)\tau_{\varepsilon}\rho}{c_{A}(\tau_{x} + \tau_{\varepsilon})^{2}} - 1}\right)\right) & \text{if } \psi < \psi^{*} \end{cases}$$

(ii) The informed investor's demand is given by

$$D_1(x^R) = \frac{q(x^R) - P_0}{\rho} (\tau_x + \tau_{\varepsilon})$$

where $q(x^R) = x^R - b(x^R)$ and

$$b\left(x^{R}\right) = \begin{cases} \frac{c_{A}(\tau_{x} + \tau_{\varepsilon})}{\rho} \left(1 - e^{-\frac{\rho}{c_{A}(\tau_{x} + \tau_{\varepsilon})}} \left(x^{R} - x^{*}\right)\right) & \text{if } x^{R} \ge x^{*} \\ -\frac{c_{A}(\tau_{x} + \tau_{\varepsilon})}{\rho} \left(1 - e^{-\frac{\rho}{c_{A}(\tau_{x} + \tau_{\varepsilon})}} \left(x^{*} - x^{R}\right)\right) & \text{if } x^{R} < x^{*} \end{cases}$$

¹⁶The case of $\psi < \psi^*$ yields a corresponding first order condition (see Appendix).

 $^{^{17}}$ Pr $oductLog\left(x\right)$ (also known as Lambert-W or Omega function) is the solution to the differential equation $f'\left(x\right) = \frac{f(x)}{x(1+f(x))}$. For a graph of the Pr oductLog function, see the Appendix.

As mentioned above, Figure 1 illustrates the analyst's forecast and his bias as a function of his private signal ψ . The parameter values used for the illustration are: $c_A = \rho = \tau_x = \tau_{\varepsilon} = S = 1$; $\mu_x = 2$.

First, notice that $b(\psi^*) = 0$. The intuition is that in a fully separating equilibrium type ψ^* is the "lowest" type in the sense that type ψ^* does not generate any trading commissions. Therefore type ψ^* achieves his best outcome if he minimizes his expected cost from forecast errors and chooses zero bias. Given that type ψ^* does not bias his forecast, his marginal expected cost of biasing his forecast are zero as well. For that to be an equilibrium, it must be that his marginal benefit from biasing his forecast are also zero. This is the case only if the informed investor attributes a marginal change of x^R , for x^R close to $x^R(\psi^*)$, almost entirely to a change in the forecast bias rather than to a change in the analyst's private signal. Indeed, the equilibrium bias function has an infinite slope at ψ^* , consistent with the investor's beliefs being insensitive to changes in the forecast at $x^R(\psi^*)$. The fact that investor's beliefs are insensitive to changes in the forecast at $x^R(\psi^*) = x^*$ can be seen in figure 2.

Since the bias is increasing in ψ in the fully separating equilibrium, the investor never attributes an increase in the forecast just to an increase in the private signal but also to an increase in the forecast bias. As a result, the sensitivity of his expectations $E\left[\widetilde{x}|x^R\right]$ to the analyst's forecast is always lower than one. This guarantees that the slope of the demand as a function of the analyst forecast is less than $\frac{1}{\rho Var(\widetilde{x}|\psi)}$. Hence, the analyst's marginal benefit from biasing his forecast is bounded. In equilibrium, the analyst's marginal cost from biasing his forecast equals his marginal benefit from doing so. Therefore, his marginal cost must be bounded as well. Since the marginal cost from biasing are linearly increasing in the bias, the bias itself must be bounded from both below and above. Figure 1 illustrates that the bias approaches its upper and lower limit asymptotically as the analyst's private signal ψ approaches $+\infty$ and $-\infty$, respectively. The consequence of an (almost) constant bias can be seen in the tails of the investor's demand. As the bias approaches a constant, the investor attributes a change in the forecast almost entirely to a change in the analyst's private signal, which results in his demand converging to the maximum slope of $\frac{1}{\rho Var(\widetilde{x}|\psi)}$ (see figure 2).

5 Empirical predictions

In practice, analysts' are subject to various kinds of incentives. These incentives may arise from trading commissions as well as other activities such as investment banking, access to senior management of the companies covered or career concerns. Any specific set of incentives an analyst is exposed to will affect the properties of his forecast. The empirical predictions in this section focus on the properties of analyst's bias that are due to trading commissions alone.

For simplicity of the analysis and disposition, we assume that there is no short-sale constraint, and that the analyst's objective function is common knowledge. However, we do not require the cost function to be quadratic and we vallow for the investor's trading cost, c_I , to be positive.

The first prediction of the model is that the analyst biases his forecast upwards more often than downwards and that when his cost function is symmetric his forecast is on average optimistic.

Corollary 1 The median bias in the analyst's forecast is positive. If the analyst's cost function is symmetric, the expected bias in his forecast is positive. ¹⁸

The analyst's incentives to bias his forecast arise from the trade he can generate. These incentives and the equilibrium bias are symmetric relative to his private signal ψ^* that generates no trade. Since the analyst's forecast reduces the uncertainty the investor is exposed to, no trade occurs for a signal that lowers the investor's expectation about the firm's earnings, i.e. $\psi^* < \mu_x$. This implies that while the bias is symmetric, it is not symmetric relative to the mid-point of the distribution but rather relative to a point that lies to the left of the prior expectation μ_x . As a result, the average bias is upwards and the average forecast error, $FE = x - x^R$, is negative. (All formal proofs are included in the Appendix.)

The prediction that the forecast is optimistic on average is supported by most of the empirical studies on analysts' earnings forecasts (see the survey in Kothari 2001). Jackson

¹⁸For c_I sufficiently high, such that μ_x is included in the "no trade zone," the median bias equals zero. In addition, if the cost function is symmetric both the mean and median forecast errors are negative (optimistic) where $FE = x - x^R$.

(2005) documents that analysts trade off accuracy for optimistic bias. The literature also offers other explanations for optimistic analyst forecasts and stock recommendations. The following is a representative but not exhaustive list. Dugar and Nathan (1995), Lin and McNichols (1998), Michaely and Womack (1999) provide evidence for affiliated analysts being more optimistic than unaffiliated analysts. Lim (2001), suggests that analyst's rationally bias their forecast upwards in order to obtain better information from the firm's management in the future. Hong and Kubik (2003) show that career concerns also may induce overoptimistic forecasts.

In our model, the extent to which the analyst's forecast is on average optimistic depends on the trading commission he can generate. Higher trading commissions per share provide the analyst with stronger incentives to bias his forecast. As a result, the magnitude of his bias is increasing in the trading commission per share. If the cost function is symmetric, an increase in the per share trading commission, c_A , not only increases the magnitude of the bias for each ψ but also increases the expected bias $E\left[b\left(\widetilde{\psi}\right)\right]$. As one would expect, as the commission per share converges to zero, the analyst's incentive to bias his forecast diminishes and his bias converges to zero for any ψ . The following Corollary summarizes these predictions.

Corollary 2 In the equilibrium of Proposition 2,

- (a) for any $\psi \not\in \Psi_{NT}$ the absolute value of $b\left(\widetilde{\psi}\right)$ is increasing in c_A ;
- (b) the expected squared forecast error, $E[FE^2]$, is increasing in c_A . If the analyst's cost function is symmetric, his expected bias, $E\left[b\left(\widetilde{\psi}\right)\right]$ is increasing in c_A ; and
- (c) for any ψ , $\lim_{c_A \to 0} b(\psi) = 0$.

The analyst's utility function depends not only on the trading commission per share, c_A , but also on the informed investor's risk aversion ρ . In particular, an increase in informed investor's risk aversion coefficient, ρ , has the exact opposite effect to an increase in c_A . This is reflected in the analyst's utility function and his equilibrium bias being a function of the ratio $\frac{c_A}{\rho}$ (see for example $b(\psi)$ in Proposition 3). A decrease in the informed investor's risk aversion increases the sensitivity of his demand to changes in his beliefs about the analyst's

private information. As a result, it also increases the sensitivity of the analyst's trading commissions to changes in the investor's beliefs about the analyst's private information, similar to the effect of an increase in c_A .

Note that if the change in the risk aversion coefficient applies not only to the informed investor but to all investors, it has an additional effect on the stock price P_0 . This change in P_0 might have an opposite effect on the analyst's incentives compared to the change in the informed investor's risk aversion. We cannot determine the overall effect of a change in all the investors' risk aversion coefficient on the analyst's equilibrium bias.

In the model, both the analyst's and the investor's equilibrium beliefs are fully rational in the sense that they are consistent with Bayes' Rule. Nevertheless, the equilibrium forecasting behavior resembles the behavior of an analyst that overweights his private information for most realizations of his private signal and truthfully reports his posterior (overweighted) expectations. In the model, the analyst acts as if he underweights his private information only if his private information falls within the interval of the firm's stock price P_0 and the prior expectations of firm's earnings μ_x . Otherwise, the analyst acts as if he overweights his private information.

Corollary 3 For all $\psi \notin [P_0, \mu_x]$ the analyst issues a forecast as if he overweights his private information ψ . That is there exists $w \geq \frac{\tau_{\varepsilon}}{\tau_x + \tau_{\varepsilon}}$ such that $x^R(\psi) = w\psi + (1 - w)\mu_x$. For all $\psi \in [P_0, \mu_x]$ the analyst issues a forecast as if he underweights his private information ψ , i.e. $w \leq \frac{\tau_{\varepsilon}}{\tau_x + \tau_{\varepsilon}}$. The inequalities hold strictly for $\psi \notin \Psi_{NT}$.

Empirical evidence suggests that analysts overweight their private information when forming their forecast. Chen and Jiang (2006) provides evidence that analysts place larger weight on their private information than we would expect them to do if they followed Bayes' Rule and truthfully reported their posterior expectation (i.e. they overweight their private information). They find that analysts that issue earnings forecasts that are higher than the consensus overweight their private information more than if they issue forecasts that are lower than the consensus they sometimes even underweight their private information. These findings are highly consistent with the predictions of our model. In particular, our model predicts that the analyst always

acts as if he overweights his private information when his private signal exceeds expectations $(\psi > \mu_x)$. When his private signal is less than expectations $(\psi < \mu_x)$, the analyst acts as if he either over- or underweight his private information. For $\psi \in [P_0, \mu_x]$ the analyst underweights his private signal (i.e., $w \leq \frac{\tau_{\varepsilon}}{\tau_x + \tau_{\varepsilon}}$). For $\psi < P_0$ the analyst overweights his private signal, as he does for $\psi > \mu_x$. However, if the analyst's cost function is symmetric, the weight he assigns to a negative surprise is lower than the weight he assigns to a positive surprise of the same magnitude (where the surprise is measured as $\psi - \mu_x$).

Chen and Jiang (2006) also finds that the deviation from Bayesian weights increases when the analysts' benefits form doing so are high or when the cost of doing so are low. This empirical finding is also in line with predictions of our model in that the analyst is fully rational and his incentives determine the extent to which he acts as if he overweights/underweights his private information.

Friesen and Weller (2006) also study the bias in analysts' earnings forecasts. ticular, they develop two models of analysts' earnings forecasts contrasting properties of unbiased rational forecasts and forecasts of analysts who are overconfident about the precision of their own information (cognitive bias). Their findings show that analysts are overconfident and do not act as if they rationally update their beliefs and truthfully report their expectations. This is consistent with our model's prediction that truth-telling cannot be sustained in equilibrium. Similar to the findings in Chen and Jiang (2006) and Friesen and Weller (2006), Easterwood and Nutt (1999) find that analysts do not act as if they rationally update their beliefs and disclose their expectations without any bias. trast, Easterwood and Nutt (1999) find that analysts overreact to positive information and underreact to negative information leading to a systematically optimistic forecasts. While our model predicts overreaction to positive information it predicts underreaction only to slightly negative news ($\psi \in [P_0, \mu_x]$). For extremely negative news our model's prediction is opposite to Easterwood and Nutt's findings. Since Easterwood and Nutt (1999) do not partition their tests according to the magnitude of the negative news, their findings not necessarily contradict our model's predictions. We are not aware of any empirical study that differentiates analysts' over/underreaction for small and large negative news.

Corollary 2 highlights the effect of the analyst's incentives on his equilibrium forecasting strategy. However, the analyst's equilibrium forecast bias does not only depend on the

trading commission per share and his cost from biasing his forecast but also on the precision of his private information. In the model, the precision of his private information, τ_{ε} , is given exogenously. In practice, analysts will typically differ in the precision of their private information due to various reasons, e.g., differences in ability, experience, general resources or access to management.

For any $\psi \notin \Psi_{NT}$ higher precision of the analyst's private information induces a forecast bias with a higher absolute value. The intuition is that the informed investor's demand is more sensitive to the information conveyed in the analyst's forecast.¹⁹ This provides the analyst with stronger incentives to bias his forecast, similar to higher trading commission, c_A .

Corollary 4 For any
$$\psi \notin \Psi_{NT}$$
, if $g'\left(x^R - x\right)$ is weakly convex in x^R then $\frac{\partial |b(\psi)|}{\partial \tau_{\varepsilon}} > 0$.

One might expect that analysts whose private information is of higher precision also obtain lower squared forecast errors on average. This would unambiguously be the case if the analyst refrained from biasing his forecast. However, since the analyst does bias his forecast, and the magnitude of the forecast is increasing in the precision of his private information, the effect of higher precision on the analyst's expected squared forecast error is ambiguous. To see this, note that $E[FE^2] = Var(\widetilde{x}|\psi) + E[b(\widetilde{\psi})^2]$. Indeed, we show that for τ_{ε} sufficiently high a further increase in the precision of the analyst's information increases rather than decreases his expected squared forecast error. We also show that for τ_{ε} sufficiently low an increase in the precision of the analyst's information decreases his expected squared forecast error for some incentive parameters. In particular, for c_A sufficiently small, the analyst's incentives to bias his forecast are relatively weak. In this case, the effect of decreasing conditional variance dominates the effect of the increased bias in the analyst's forecast, resulting in that the expected forecast error is decreasing in the precision of the analyst's private information. This holds as long as the precision of the analyst's private signal is not "too high."

Figure 6 illustrates the effect of different precisions on the analyst's expected squared forecast errors for two different set of parameters. It emphasizes that the expected squared

¹⁹Investor's demand is more sensitive to the investor's beliefs about the analyst's private information ψ because of two effects. First, the conditional expectations $E\left[\widetilde{x}|\psi\right]$ assign higher weight to the private signal ψ . Second, the demand function $D_1 = \frac{E\left[\widetilde{x}|\psi\right] - P_0}{\rho Var\left(\widetilde{x}|\psi\right)}$ is steeper since the conditional variance is lower.

forecast error can be either increasing or decreasing in the precision of the analyst's private information for lower values of τ_{ε} , but is always increasing in τ_{ε} for sufficiently high values of τ_{ε} .²⁰

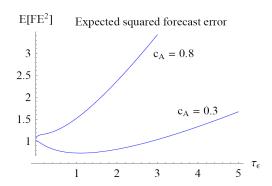


Figure 6: Expected Squared Forecast Error as a Function of τ_{ε}

The precision of the analyst's private information does not only affect the expected squared forecast error but also the expected bias of the forecast. While we were not able to show that the expected forecast error is monotonically increasing in the precision of the analyst's private information, numerical simulations suggest that this is indeed the case. The following Corollary presents the predictions that we established analytically.

Corollary 5 In the equilibrium of Proposition 2,

- (a) $\lim_{\tau_{\varepsilon}\to\infty} E[FE^2] = \infty$.
- (b) $\lim_{\tau_{\varepsilon}\to\infty} E[FE] = \infty$ for all symmetric cost functions $g(\cdot)$.
- (c) For any τ_{ε} and $\delta > 0$, there exists c_A^* such that for all $c_A < c_A^*$ $E[FE^2]|_{\tau_{\varepsilon}+\delta} < E[FE^2]|_{\tau_{\varepsilon}}$ (i.e. that the expected squared forecast error is decreasing over a certain interval of values of τ_{ε}).

²⁰The graph illustrates that the expected squared forecast error can be both increasing and decreasing. However, it does not show monotonicity of the expected squared forecast error with respect to τ_{ε} in any region.

To summarize, the model predicts that analysts with more precise private information not necessarily issue forecasts that result in smaller expected forecast bias and/or smaller expected squared forecast errors. This might shed some light on the surprising empirical findings that affiliated analysts who are conjectured to possess more precise information about a firm do not outperform independent analysts, in the sense of issuing forecasts that are less biased and result in smaller forecast error (e.g., Gu and Xue 2007).

6 Conclusion

This paper contains a model of the interaction between sell-side analysts and their client-investors. In the model, the analyst issues a forecast, based on which the investor may decide to trade. The analyst benefits from the investor's trade in the form of trading commissions he receives. Since the analyst is not confined to unbiased forecasts, his incentives from trading commissions lead the analyst to bias his forecast. In equilibrium, the analyst trades off benefits from trading commissions against cost from forecast errors. The paper identifies and analyzes the unique fully separating equilibrium of this game. It also studies short sale constraints, uncertainty about the analyst's incentives and different levels of investor's trading commissions. The paper demonstrates that the existence of the equilibrium is robust to these extensions and studies their effect on the equilibrium behavior of both the analyst and the investor.

The model provides several empirical predictions including: (i) the analyst biases his forecast upward more often than downward and his forecast is on average optimistic if the cost function is symmetric; (ii) the analyst acts as if he overweights private information if this private information is favorable. If his private information is sufficiently unfavorable, he also acts as if he overweights it, but to a lesser extent. If his private information is slightly unfavorable he acts as if he underweights his private information; and (iii) the analyst's expected squared forecast error may either increase or decrease in the precision of his private information. If the precision of his private information is sufficiently high, a further increase in the precision always increases rather than decreases (as one might expect) his expected squared forecast error. Hence, the model suggests that the relation between an analyst's squared forecast error and the precision of his private signal is not necessarily negative and

depends on the forecasting environment.

Even though part of the model's novel predictions are supported by existing empirical studies, the paper also provides additional predictions that have not yet been tested empirically. These new predictions set the ground for future empirical work that might shed additional light on the behavior of sell-side analysts and their interaction with their client-investors.

7 Appendix

7.1 Proof of $D_0 = D$

We can compute $D_1(\psi)$ as

$$D_{1}(\psi) = \begin{cases} \frac{E[\widetilde{x}|x^{R}] - P_{0} - c_{I}}{\rho/(\tau_{x} + \tau_{\varepsilon})} & \text{if } x^{R} > \psi^{*} + c_{I} \frac{\tau_{x} + \tau_{\varepsilon}}{\tau_{\varepsilon}} \\ D_{0} & \text{if } \psi^{*} + c_{I} \frac{\tau_{x} + \tau_{\varepsilon}}{\tau_{\varepsilon}} > \psi > \psi^{*} - c_{I} \frac{\tau_{x} + \tau_{\varepsilon}}{\tau_{\varepsilon}} \\ \frac{E[\widetilde{x}|x^{R}] - P_{0} + c_{I}}{\rho/(\tau_{x} + \tau_{\varepsilon})} & \text{if } \psi^{*} - c_{I} \frac{\tau_{x} + \tau_{\varepsilon}}{\tau_{\varepsilon}} > x^{R} \end{cases}$$

where

$$\psi^* = \frac{\tau_x + \tau_\varepsilon}{\tau_\varepsilon} \left(\frac{D_0 \rho}{\tau_x + \tau_\varepsilon} + P_0 - \frac{\tau_x \mu_x}{\tau_x + \tau_\varepsilon} \right). \tag{6}$$

For notational convenience let the no-trading region be $\Psi_{NT} = \left[\underline{\psi}, \overline{\psi}\right]$ where $\underline{\psi} = \psi^* - c_I \frac{\tau_x + \tau_\varepsilon}{\tau_\varepsilon}$ and $\overline{\psi} = \psi^* + c_I \frac{\tau_x + \tau_\varepsilon}{\tau_\varepsilon}$. Given the optimal $D_1(\psi)$ and D_0 we can compute the ex-ante expected utility as

$$-\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\rho[W_{0}+(\widetilde{x}-P_{0})D_{1}-c_{I}|D_{1}-D_{0}|]} f\left(x|\psi\right) g\left(\psi\right) dx d\psi$$

$$= -\int_{-\infty}^{\underline{\psi}} e^{-\rho\left[W_{0}+(E[\widetilde{x}|\psi]-P_{0})\frac{E[\widetilde{x}|\psi]-P_{0}+c_{I}}{\rho Var(\widetilde{x}|\psi)}+c_{I}\left(\frac{E[\widetilde{x}|\psi]-P_{0}+c_{I}}{\rho Var(\widetilde{x}|\psi)}-D_{0}\right)-\frac{\rho}{2}\left(\frac{E[\widetilde{x}|\psi]-P_{0}+c_{I}}{\rho Var(\widetilde{x}|\psi)}\right)^{2} Var(\widetilde{x}|\psi)\right]} g\left(\psi\right) d\psi$$

$$-\int_{\underline{\psi}}^{\overline{\psi}} e^{-\rho\left[W_{0}+(E[\widetilde{x}|\psi]-P_{0})D_{0}-\frac{\rho}{2}D_{0}^{2}Var(\widetilde{x}|\psi)\right]} g\left(\psi\right) d\psi$$

$$-\int_{\underline{\psi}}^{\infty} e^{-\rho\left[W_{0}+(E[\widetilde{x}|\psi]-P_{0})\frac{E[\widetilde{x}|\psi]-P_{0}-c_{I}}{\rho Var(\widetilde{x}|\psi)}-c_{I}\left(\frac{E[\widetilde{x}|\psi]-P_{0}-c_{I}}{\rho Var(\widetilde{x}|\psi)}-D_{0}\right)-\frac{\rho}{2}\left(\frac{E[\widetilde{x}|\psi]-P_{0}-c_{I}}{\rho Var(\widetilde{x}|\psi)}\right)^{2} Var(\widetilde{x}|\psi)\right]} g\left(\psi\right) d\psi$$

The informed investor chooses D_0 that maximizes his ex-ante expected utility taking into account his optimal trading strategy at t = 1. Equation (6) shows a one-to-one relation between ψ^* and D_0 . Hence, solving for the optimal D_0 is equivalent to solving for the optimal ψ^* . One can show that the informed investor's objective is to choose ψ^* that maximizes the following expression.

$$\max_{\psi^*} k \int_{-\infty}^{\infty} \delta(\psi, \psi^*) \, \phi(\psi) \, d\psi \tag{7}$$

where k is a positive constant, $\phi(\psi)$ denotes the pdf of a normal distribution with mean P_0 and precision τ_{ε} and

$$\delta\left(\psi,\psi^{*}\right) = \begin{cases} -e^{cI} \frac{E\left[\overline{x}\left|\psi^{*}\right] - E\left[\overline{x}\left|\psi\right] - \frac{1}{2}c_{I}}{Var\left(\overline{x}\left|\psi\right)} & \text{if } \psi < \psi \\ -e^{\frac{\left(E\left[\overline{x}\left|\psi\right] - E\left[\overline{x}\left|\psi^{*}\right]\right)^{2}}{2Var\left(\overline{x}\left|\psi\right)}} & \text{if } \underline{\psi} \leq \psi \leq \overline{\psi} \\ -e^{cI} \frac{E\left[\overline{x}\left|\psi\right] - E\left[\overline{x}\left|\psi^{*}\right] - \frac{1}{2}c_{I}}{Var\left(\overline{x}\left|\psi\right)} & \text{if } \overline{\psi} < \psi \end{cases}.$$

 $\delta(\psi, \psi^*)$ is continuous, symmetric around ψ^* with a unique global maximum at $\psi = \psi^*$. From this it follows that the ψ^* that maximizes (7) equals P_0 . Solving (3) for $\psi^* = P_0$ yields $D_0 = D = S$.

7.2 Proof of Proposition 1

The analyst's optimization problem is

$$\max_{x^R} u^A (x^R, \psi) = c_A |D_1(x^R) - D_0| - E [g(x^R - \widetilde{x}) | \psi].$$

When solving for his optimal forecast, the analyst takes the informed investor's demand as given. Since the informed investor's demand, $D_1(x^R)$, is monotonically increasing in x^R and $D_1(x^*) = D_0$ we can rewrite the analyst's optimization problem as

$$\max_{x^R} u^A \left(x^R, \psi \right) = \begin{cases} c_A \left(D_1 \left(x^R \right) - D_0 \right) - E \left[g \left(x^R - \widetilde{x} \right) \middle| \psi \right] & \text{if } x^R \ge x^* \\ c_A \left(D_0 - D_1 \left(x^R \right) \right) - E \left[g \left(x^R - \widetilde{x} \right) \middle| \psi \right] & \text{if } x^R < x^* \end{cases}$$

where the informed investor's demand is given by

$$D_1(x^R) = \frac{x^R - \widehat{b}(x^R) - P_0}{\rho Var(\widetilde{x}|\psi)}$$

and $\hat{b}(x^R)$ denotes investors' believes about the bias given the analyst's forecast x^R . Taking the informed investor's demand function, $D_1(x^R)$, and beliefs, $\hat{b}(x^R)$, as given, the analyst maximizes

$$u_{A}\left(x^{R},\psi\right) = \begin{cases} c_{A}\left(\frac{x^{R}-\widehat{b}\left(x^{R}\right)-P_{0}}{\rho Var\left(\widetilde{x}|\psi\right)} - D_{0}\right) - E\left[g\left(x^{R}-\widetilde{x}\right)|\psi\right] & \text{if } x^{R} \geq x^{*} \\ c_{A}\left(D_{0} - \frac{x^{R}-\widehat{b}\left(x^{R}\right)-P_{0}}{\rho Var\left(\widetilde{x}|\psi\right)}\right) - E\left[g\left(x^{R}-\widetilde{x}\right)|\psi\right] & \text{if } x^{R} < x^{*} \end{cases}$$

Then, the (local) FOCs to the analyst's optimization problem are given by

$$\begin{cases}
c_A \frac{1 - \frac{\partial \widehat{b}(x^R)}{\partial x^R}}{\rho Var(\widehat{x}|\psi)} - \frac{\partial}{\partial x^R} E\left[g\left(x^R - \widetilde{x}\right)|\psi\right] = 0 & \text{if } x^R > x^* \\
-c_A \frac{1 - \frac{\partial \widehat{b}(x^R)}{\partial x^R}}{\rho Var(\widehat{x}|\psi)} - \frac{\partial}{\partial x^R} E\left[g\left(x^R - \widetilde{x}\right)|\psi\right] = 0 & \text{if } x^R < x^*
\end{cases}$$

In equilibrium we need that

$$x^{R}(\psi) = E\left[\widetilde{x}|\psi\right] + \widehat{b}\left(x^{R}\right)$$

With that,

$$\frac{\partial \widehat{b}\left(x^{R}\right)}{\partial x^{R}} = \frac{\partial b\left(\psi\right)}{\partial \psi} \frac{1}{\frac{\partial x^{R}}{\partial \psi}} = b'\left(\psi\right) \frac{1}{\frac{\partial E\left[\widetilde{x}\left|\psi\right|\right]}{\partial \psi} + b'\left(\psi\right)}$$

where $\frac{\partial E[\widetilde{x}|\psi]}{\partial \psi} = \frac{\tau_{\varepsilon}}{\tau_{x} + \tau_{\varepsilon}}$. With that we can rewrite the FOC as a function of $b(\psi)$

$$\begin{cases}
c_{A} \frac{1-b'(\psi)\frac{1}{\frac{\partial E[\widetilde{x}]\psi]}+b'(\psi)}{\rho Var(\widetilde{x}]\psi)} - \frac{\partial}{\partial b} E\left[g\left(E\left[\widetilde{x}|\psi\right]+b-\widetilde{x}\right)|\psi\right] = 0 & \text{if } \psi > P_{0} \\
-c_{A} \frac{1-b'(\psi)\frac{1}{\frac{\partial E[\widetilde{x}]\psi]}+b'(\psi)}{\rho Var(\widetilde{x}|\psi)} - \frac{\partial}{\partial b} E\left[g\left(E\left[\widetilde{x}|\psi\right]+b-\widetilde{x}\right)|\psi\right] = 0 & P_{0} > \psi
\end{cases}$$
(8)

(substituting $b(x^R)$ for $\widehat{b}(x^R)$)

$$\begin{cases}
c_A \frac{1 - b'(x^R)}{\rho Var(\widetilde{x}|\psi)} - \frac{\partial}{\partial b} E\left[g\left(E\left[\widetilde{x}|\psi\right] + b - \widetilde{x}\right)|\psi\right] = 0 & \text{if } x^R > x^* \\
-c_A \frac{1 - b'(x^R)}{\rho Var(\widetilde{x}|\psi)} - \frac{\partial}{\partial b} E\left[g\left(E\left[\widetilde{x}|\psi\right] + b - \widetilde{x}\right)|\psi\right] = 0 & \text{if } x^R < x^*
\end{cases} \tag{9}$$

Solving for $b'(x^R)$ yields

$$\begin{cases} b'\left(x^{R}\right) = 1 - \frac{\rho Var(\widetilde{x}|\psi)}{c_{A}} \frac{\partial}{\partial b} E\left[g\left(E\left[\widetilde{x}|\psi\right] + b - \widetilde{x}\right)|\psi\right] & \text{if } x^{R} > x^{*} \\ b'\left(x^{R}\right) = 1 - \frac{\rho Var(\widetilde{x}|\psi)}{c_{A}} \frac{\partial}{\partial b} E\left[g\left(E\left[\widetilde{x}|\psi\right] + b - \widetilde{x}\right)|\psi\right] & \text{if } x^{R} < x^{*} \end{cases}$$

We know that for $x^R = x^*$ the analyst does not bias his forecast, i.e. $\frac{\partial}{\partial b} E\left[g\left(E\left[\widetilde{x}|\psi\right] + b - \widetilde{x}\right)|\psi\right] = 0$. Hence, the RHS is continuous in x^R at $x^R = x^*$. From the Fundamental Theorem of Differential Equations it follows that there exists a solution $b\left(x^R\right)$. Since $b\left(x^R\right)$ is bounded and continuous there always exists a $b\left(\psi\right)$ that solves the following equation.

$$b(\psi) - b(x^R | x^R = E[\widetilde{x} | \psi] + b(\psi)) = 0$$

Since there is a solution to the FOC which includes consistent investor beliefs, there exists an equilibrium.

7.3 Proof of Proposition 2

The informed investor's certainty equivalent given the forecast x^R is

$$E[U(D_1)|x^R] = W_0 + D_1(E[\widetilde{x}|x^R] - P_0) - c_I|D_1 - D_0| - \frac{\rho}{2}D_1^2 Var(\widetilde{x}|x^R).$$

The FOC for x^R sufficiently high such that trade occurs yields

$$E\left[\widetilde{x}|x^{R}\right] - P_{0} - c_{I} - \rho D_{1} \frac{1}{\tau_{x} + \tau_{\varepsilon}} = 0$$

$$\frac{E\left[\widetilde{x}|x^{R}\right] - P_{0} - c_{I}}{\rho Var\left(\widetilde{x}|x^{R}\right)} = D_{1}$$

The highest signal x^R for which no trade occurs is given by $D_1 = D_0$, where $D_0 = \frac{\mu_x - P_0}{\rho} \tau_x$. Let $\widehat{q}(x^R)$ denote investors' beliefs about $E\left[\widetilde{x}|x^R\right]$. Suppose $\widehat{q}(x^R)$ is strictly increasing in x^R (due to the full separation of the equilibrium). Note that $Var\left(\widetilde{x}|x^R\right) = Var\left(\widetilde{x}|\psi\right)$. This holds because $Var\left(\widetilde{x}|\psi\right)$ does not depend on the realization ψ and the equilibrium is fully separating.

$$\frac{\widehat{q}(x^R) - P_0 - c_I}{\rho Var(\widetilde{x}|x^R)} = \frac{\mu_x - P_0}{\rho} \tau_x$$

$$\widehat{q}(x^R) = \frac{\mu_x \tau_x + P_0 \tau_\varepsilon}{\tau_x + \tau_\varepsilon} + c_I$$

So, $x^R = \widehat{q}^{-1} \left(P_0 + c_I \frac{\tau_x + \tau_\varepsilon}{\tau_x} \right)$ is the highest forecast for which there will be no trade. Then, investors' demand function is given by

$$D_{1}\left(x^{R}\right) = \begin{cases} \frac{\widehat{q}\left(x^{R}\right) - P_{0} - c_{I}}{\rho Var\left(\widetilde{x}|x^{R}\right)} & \text{if } \widehat{q}\left(x^{R}\right) > \frac{\mu_{x}\tau_{x} + P_{0}\tau_{\varepsilon}}{\tau_{x} + \tau_{\varepsilon}} + c_{I} \\ D_{0} & \text{if } \frac{\mu_{x}\tau_{x} + P_{0}\tau_{\varepsilon}}{\tau_{x} + \tau_{\varepsilon}} + c_{I} > \widehat{q}\left(x^{R}\right) > \frac{\mu_{x}\tau_{x} + P_{0}\tau_{\varepsilon}}{\tau_{x} + \tau_{\varepsilon}} - c_{I} \\ \frac{\widehat{q}\left(x^{R}\right) - P_{0} + c_{I}}{\rho Var\left(\widetilde{x}|x^{R}\right)} & \text{if } \frac{\mu_{x}\tau_{x} + P_{0}\tau_{\varepsilon}}{\tau_{x} + \tau_{\varepsilon}} - c_{I} > \widehat{q}\left(x^{R}\right) \end{cases}$$

Taking investors' demand function as given, the analyst's objective function $U_A(\psi, x^R)$ equals

$$\begin{cases}
c_{A}\left(\frac{\widehat{q}(x^{R})-P_{0}-c_{I}}{\rho Var(\widetilde{x}|x^{R})}-D_{0}\right)-E\left[g\left(x^{R}-\widetilde{x}\right)|\psi\right] & \text{if } \widehat{q}\left(x^{R}\right)>\frac{\mu_{x}\tau_{x}+P_{0}\tau_{\varepsilon}}{\tau_{x}+\tau_{\varepsilon}}+c_{I} \\
-E\left[g\left(x^{R}-\widetilde{x}\right)|\psi\right] & \text{if } \frac{\mu_{x}\tau_{x}+P_{0}\tau_{\varepsilon}}{\tau_{x}+\tau_{\varepsilon}}+c_{I}>\widehat{q}\left(x^{R}\right)>\frac{\mu_{x}\tau_{x}+P_{0}\tau_{\varepsilon}}{\tau_{x}+\tau_{\varepsilon}}-c_{I} \\
-c_{A}\left(\frac{\widehat{q}(x^{R})-P_{0}+c_{I}}{\rho Var(\widetilde{x}|x^{R})}-D_{0}\right)-E\left[g\left(x^{R}-\widetilde{x}\right)|\psi\right] & \text{if } \frac{\mu_{x}\tau_{x}+P_{0}\tau_{\varepsilon}}{\tau_{x}+\tau_{\varepsilon}}-c_{I}>\widehat{q}\left(x^{R}\right)
\end{cases}$$

Taking investors' beliefs $\widehat{q}\left(x^{R}\right)=x^{R}-\widehat{b}\left(x^{R}\right)$ as given, the analyst's optimal forecast is given by

$$\begin{cases} c_{A} \frac{1 - \frac{d\widehat{b}\left(x^{R}\right)}{dx^{R}}}{\rho Var(\widehat{x}|x^{R})} - \frac{\partial}{\partial x^{R}} E\left[g\left(x^{R} - \widetilde{x}\right) \middle| \psi\right] = 0 & \text{if } \widehat{q}\left(x^{R}\right) > \frac{\mu_{x}\tau_{x} + P_{0}\tau_{\varepsilon}}{\tau_{x} + \tau_{\varepsilon}} + c_{I} \\ \frac{\partial}{\partial x^{R}} E\left[g\left(x^{R} - \widetilde{x}\right) \middle| \psi\right] = 0 & \text{if } \frac{\mu_{x}\tau_{x} + P_{0}\tau_{\varepsilon}}{\tau_{x} + \tau_{\varepsilon}} + c_{I} > \widehat{q}\left(x^{R}\right) > \frac{\mu_{x}\tau_{x} + P_{0}\tau_{\varepsilon}}{\tau_{x} + \tau_{\varepsilon}} - c_{I} \\ -c_{A} \frac{1 - \frac{d\widehat{b}\left(x^{R}\right)}{\rho Var(\widetilde{x}|x^{R})}}{\rho Var(\widetilde{x}|x^{R})} - \frac{\partial}{\partial x^{R}} E\left[g\left(x^{R} - \widetilde{x}\right) \middle| \psi\right] = 0 & \text{if } \frac{\mu_{x}\tau_{x} + P_{0}\tau_{\varepsilon}}{\tau_{x} + \tau_{\varepsilon}} - c_{I} > \widehat{q}\left(x^{R}\right) \end{cases}$$

imposing the equilibrium condition that investors' conjecture coincides with analysts' forecasting strategy

$$\widehat{q}\left(x^{R}\left(\psi\right)\right) = E\left[\widetilde{x}|\psi\right]$$

This implies that

$$\begin{split} \widehat{q}\left(x^{R}\left(\psi\right)\right) &> \frac{\mu_{x}\tau_{x} + P_{0}\tau_{\varepsilon}}{\tau_{x} + \tau_{\varepsilon}} + c_{I} \Leftrightarrow \psi > P_{0} + c_{I}\frac{\tau_{x} + \tau_{\varepsilon}}{\tau_{e}} \\ \widehat{q}\left(x^{R}\left(\psi\right)\right) &< \frac{\mu_{x}\tau_{x} + P_{0}\tau_{\varepsilon}}{\tau_{x} + \tau_{\varepsilon}} - c_{I} \Leftrightarrow \psi < P_{0} - c_{I}\frac{\tau_{x} + \tau_{\varepsilon}}{\tau_{e}} \end{split}$$

With this we can rewrite the analyst's FOC as

$$\left\{ \begin{array}{ll} c_A \frac{1 - b'(\psi) \frac{1}{\partial E\left[\widetilde{x}|\psi\right] + b'(\psi)}}{\rho Var(\widetilde{x}|\psi)} - \frac{\partial}{\partial b} E\left[g\left(E\left[\widetilde{x}|\psi\right] + b - \widetilde{x}\right)|\psi\right] = 0 & \text{if } \psi > P_0 + c_I \frac{\tau_x + \tau_\varepsilon}{\tau_\varepsilon} \\ \frac{\partial}{\partial b} E\left[g\left(E\left[\widetilde{x}|\psi\right] + b - \widetilde{x}\right)|\psi\right] = 0 & P_0 + c_I \frac{\tau_x + \tau_\varepsilon}{\tau_\varepsilon} > \psi > P_0 - c_I \frac{\tau_x + \tau_\varepsilon}{\tau_\varepsilon} \\ - c_A \frac{1 - b'(\psi) \frac{1}{\partial E\left[\widetilde{x}|\psi\right] + b'(\psi)}}{\rho Var(\widetilde{x}|\psi)} - \frac{\partial}{\partial b} E\left[g\left(E\left[\widetilde{x}|\psi\right] + b - \widetilde{x}\right)|\psi\right] = 0 & P_0 - c_I \frac{\tau_x + \tau_\varepsilon}{\tau_\varepsilon} > \psi \end{array} \right.$$

The FOC of the first and third case are identical to the differential equation that we solved above (see equation 8). So, the only difference is the boundary condition $b\left(P_0 + c_I \frac{\tau_x + \tau_\varepsilon}{\tau_\varepsilon}\right) = 0$ for the right bound and $b\left(P_0 - c_I \frac{\tau_x + \tau_\varepsilon}{\tau_\varepsilon}\right) = 0$ for the left bound. We know that type $\psi = P_0 + c_I \frac{\tau_x + \tau_\varepsilon}{\tau_\varepsilon}$ does not want to mimic any type higher than $P_0 + c_I \frac{\tau_x + \tau_\varepsilon}{\tau_\varepsilon}$. As a result, any type $\psi < P_0 + c_I \frac{\tau_x + \tau_\varepsilon}{\tau_\varepsilon}$ also does not want to mimic any type higher than $P_0 + c_I \frac{\tau_x + \tau_\varepsilon}{\tau_\varepsilon}$ because his marginal cost of mimicing are even higher than those of type $\psi = P_0 + c_I \frac{\tau_x + \tau_\varepsilon}{\tau_\varepsilon}$ and the marginal benefits are the same.

Hence, the optimal bias function and investors' beliefs are consistent with the Proposition.

7.4 Short-Selling Restrictions

Investors' maximize their certainty equivalent subject to $D_1 \geq -s$. Here, s is the maximum number of shares an investor can short sell. As a result, the maximum number of shares he can sell is the sum of s and his initial holdings D_0 . To simplify the notation, we assume that $c_I = 0$. However, none of the arguments is qualitatively affected for $c_I > 0$.

Claim 7 If the investor faces a short sale constraint of s, there is an equilibrium defined as follows.

(i) The analyst's optimal forecasts is given by

$$x^{R}(\psi) = \begin{cases} E\left[\widetilde{x}|\psi\right] + b\left(\psi\right) & \text{if } \psi \ge \underline{\psi} \\ E\left[\widetilde{x}|\underline{\psi}\right] + b\left(\underline{\psi}\right) & \text{if } \underline{\psi} > \overline{\psi} \ge \psi' \\ E\left[\widetilde{x}|\psi\right] & \text{if } \overline{\psi} < \psi' \end{cases}$$

where $b(\psi)$ denotes the analyst's equilibrium bias in the absence of any short sale restrictions and $\underline{\psi}$ is the private signal for which the short sale constraint becomes binding, that is

$$\underline{\psi} = \frac{\left(P_0 - \frac{\rho s}{\tau_x + \tau_{\varepsilon}}\right) (\tau_x + \tau_{\varepsilon}) - \mu_x \tau_x}{\tau_{\varepsilon}}$$

Note that all types $\psi \in [\psi', \underline{\psi}]$ issue the same pooling forecast $x_{pool}^R = E[\widetilde{x}|\underline{\psi}] + b(\underline{\psi})$. All types $\psi \leq \psi'$ do not bias their forecast. ψ' is given by

$$\psi' = \underline{\psi} + \frac{\tau_x + \tau_{\varepsilon}}{\tau_{\varepsilon}} b\left(\underline{\psi}\right).$$

Note that $\underline{\psi} < \psi^*$ implies that $b\left(\underline{\psi}\right) < 0$.

(ii) The informed investor's demand is given by

$$D_1(x^R) = \begin{cases} \frac{q(x^R) - P_0}{\rho Var(\widetilde{x}|\psi)} & \text{if } x^R \ge E\left[\widetilde{x}|\underline{\psi}\right] \\ -s & \text{otherwise} \end{cases}$$

where $q(x^R)$ is investors' expectation of the firm's earnings given the analyst forecast.

$$q\left(x^{R}\right) = \begin{cases} x^{R} - b\left(x^{R}\right) & \text{if } x^{R} > x_{pool}^{R} \\ E\left[\widetilde{x}|\psi \in \left[\psi',\underline{\psi}\right]\right] & \text{if } x^{R} = x_{pool}^{R} \\ x^{R} & \text{if } x_{pool}^{R} > x^{R} \end{cases}$$

where $b\left(x^{R}\right)$ denotes investors' inferences about the analyst's bias if the forecast is x^{R} .

Proof. Note that whenever a pooling report occurs, investors' beliefs about the firm's earnings are no longer normally distributed. Hence, we cannot rely on the standard result that investors' certainty equivalent takes the mean-variance form. To resolve this problem, we first show that investors' demand, in the absence of short sale constraints, following a pooling report over the interval $[\psi', \psi]$ is increasing in ψ' . This guarantees that the demand given that $\psi \in [\psi', \psi]$ is lower than the demand given that $\psi = \underline{\psi}$. This result is not trivial since if investors learn that $\psi \in [\psi', \underline{\psi}]$ the conditional variance $Var\left(x|\psi\in[\psi',\underline{\psi}]\right)$ is higher than the conditional variance $Var\left(x|\underline{\psi}\right)$. When we compare investors' demand if they learn $\psi \in [\psi', \underline{\psi}]$ versus $\psi = \underline{\psi}$, we need to consider two opposite effects. First, the fact that $E\left[x|\psi\in[\psi',\underline{\psi}]\right] < E\left[x|\underline{\psi}\right]$ causes investors' demand following $\psi\in[\psi',\underline{\psi}]$ to be more negative/lower (holding all else constant). On the other hand, the fact that $Var\left(x|\psi\in[\psi',\underline{\psi}]\right) > Var\left(x|\underline{\psi}\right)$ causes investors' demand following $\psi\in[\psi',\underline{\psi}]$ to be less negative/higher (holding all else constant). Once, we establish the above result, the reminder of the proof is straight forward.

Investor's demand following a pooling report

Let $f(\widetilde{x}|\Omega)$ denote the conditional probability distribution of \widetilde{x} given investors' information set Ω .

$$\max_{D_1 \ge -s} E\left[U\left(D_1\right) | x^R\right] = -\int_{-\infty}^{\infty} e^{-\rho(W_o + D_1(\widetilde{x} - P_0))} f\left(\widetilde{x} | x^R\right) dx$$

For notational convenience let

$$k(x, D_1) = e^{-\rho(W_o + D_1(\widetilde{x} - P_0))} (\widetilde{x} - P_0)$$

First, we solve the unconstrained optimization problem in the absence of any short sale restriction. There is always an interior solution D_1^u to the unconstrained optimization problem. The FOC yields

$$\int_{-\infty}^{\infty} k(\widetilde{x}, D_1^u) f(\widetilde{x}|x^R) dx = 0$$

Following a separating forecast, where investor learns that $\{\psi | (x^R(\psi) = x^R)\} = \psi$, we have

$$\int_{-\infty}^{\infty} k(\widetilde{x}, D_1^u) f(\widetilde{x}|\psi) dx = 0$$

For notational convenience let

$$K(\psi) = \int_{-\infty}^{\infty} k(\widetilde{x}, D_1) f(\widetilde{x}|\psi) dx$$

We know that $D_1^u(\psi)$ is strictly increasing in ψ . We also know that

$$\frac{\partial K\left(\psi\right)}{\partial D_{1}} = -\rho \int_{-\infty}^{\infty} e^{-\rho(W_{o} + D_{1}(\widetilde{x} - P_{0}))} \left(\widetilde{x} - P_{0}\right)^{2} f\left(\widetilde{x}|\psi\right) dx < 0$$

From the Implicit Function Theorem it follows that

$$\frac{\partial D_1^u(\psi)}{\partial \psi} = -\frac{\frac{\partial K(\psi)}{\partial \psi}}{\frac{\partial K(\psi)}{\partial D_1}} > 0$$

and hence

$$\frac{\partial K\left(\psi\right)}{\partial \psi} > 0.$$

Following a pooling forecast, where the investor learns that $\{\psi|x^R(\psi)=x^R\}=[\psi',\underline{\psi}]$, we have

$$\int_{-\infty}^{\infty} k(\widetilde{x}, D_1^u) f(\widetilde{x}|\psi \in [\psi', \underline{\psi}]) dx = 0$$

Applying the law of iterated expectations, we can rewrite the FOC as

$$\int_{\psi'}^{\underline{\psi}} \left(\int_{-\infty}^{\infty} k\left(\widetilde{x}, D_1^u\right) f\left(\widetilde{x}|\psi\right) dx \right) h\left(\widetilde{\psi} \middle| \widetilde{\psi} \in \left[\psi', \underline{\psi}\right] \right) d\psi = 0 \tag{10}$$

where $h(\cdot)$ denotes the (conditional) probability density function of ψ . We want to show that $D_1^u \left(\psi \in \left[\psi', \underline{\psi} \right] \right)$ is increasing in ψ' . With $K(\psi) = \int_{-\infty}^{\infty} k\left(\widetilde{x}, D_1^u \right) f\left(\widetilde{x} | \psi \right) dx$ we can rewrite (10) as

$$\int_{\psi'}^{\underline{\psi}} K(\psi) h\left(\widetilde{\psi} \middle| \widetilde{\psi} \in \left[\psi', \underline{\psi}\right]\right) d\psi = 0$$

We know that

$$\frac{\partial}{\partial D_{1}} \int_{\psi'}^{\underline{\psi}} K\left(\psi\right) h\left(\widetilde{\psi} \middle| \widetilde{\psi} \in \left[\psi', \underline{\psi}\right]\right) d\psi = \int_{\psi'}^{\underline{\psi}} \frac{\partial K\left(\psi\right)}{\partial D_{1}} h\left(\widetilde{\psi} \middle| \widetilde{\psi} \in \left[\psi', \underline{\psi}\right]\right) d\psi < 0.$$

If we can show that

$$\frac{\partial}{\partial \psi'} \int_{\psi'}^{\underline{\psi}} K(\psi) h\left(\widetilde{\psi} \middle| \widetilde{\psi} \in \left[\psi', \underline{\psi}\right]\right) d\psi > 0 \tag{11}$$

then it follows from the Implicit Function Theorem that D_1^u ($\psi \in [\psi', \underline{\psi}]$) is increasing in ψ' . We can rewrite (11) as

$$\frac{\partial}{\partial \psi'} \int_{\psi'}^{\underline{\psi}} K(\psi) \frac{h(\psi)}{H(\psi) - H(\psi')} d\psi > 0$$

where $h(\cdot)$ and $H(\cdot)$ denote the pdf and cdf of ψ respectively.

$$\frac{\partial}{\partial \psi'} \int_{\psi'}^{\underline{\psi}} K(\psi) \frac{h(\psi)}{H(\underline{\psi}) - H(\psi')} d\psi = \int_{\psi'}^{\underline{\psi}} K(\psi) \frac{h(\psi)h(\psi')}{\left(H(\underline{\psi}) - H(\psi')\right)^2} d\psi - K(\psi') \frac{h(\psi')}{H(\underline{\psi}) - H(\psi')}$$

$$= \frac{h(\psi')}{H(\underline{\psi}) - H(\psi')} \left(\int_{\psi'}^{\underline{\psi}} K(\psi) \frac{h(\psi)}{H(\underline{\psi}) - H(\psi')} d\psi - K(\psi')\right)$$

$$= \frac{h(\psi')}{H(\underline{\psi}) - H(\psi')} \int_{\psi'}^{\underline{\psi}} (K(\psi) - K(\psi')) \frac{h(\psi)}{H(\underline{\psi}) - H(\psi')} d\psi > 0$$

which follows from $K(\psi)$ being increasing in ψ .

So, if the investor learns that $\psi \in [\psi', \underline{\psi}]$ for any $\psi' < \underline{\psi}$ (which is the case following a pooling report) then his demand satisfies the following

$$D_1^u\left(\left[\psi',\underline{\psi}\right]\right) < D_1^u\left(\underline{\psi}\right)$$

This implies that the informed investor's demand following the pooling forecast (where the pooling interval can be of any size) is lower than his demand if he learns that $\psi = \underline{\psi}$.

Equilibrium forecasting strategy

In this part we show that the analyst does not want to deviate from the forecasting strategy

in the Claim.

For any $\psi > \underline{\psi}$ the equilibrium forecast and the investor's beliefs are exactly the same as in the equilibrium without any short sale constraints. This guarantees that none of the types $\psi > \underline{\psi}$ will deviate to any other $x^R > x^R(\underline{\psi})$. In addition, none of the types $\psi > \underline{\psi}$ will deviate to a report $x^R \leq x^R(\underline{\psi})$. To see that, note that all reports $x^R < x^R(\underline{\psi})$ will not generate more trade than $x^R(\underline{\psi})$ but will induce higher expected cost from forecasting errors.

For any $\psi \in [\psi', \underline{\psi}]$ the equilibrium forecast is x_{pool}^R . Clearly, an analyst that observes any $\psi \in [\psi', \underline{\psi}]$ would never issue a forecast that is lower than x_{pool}^R . The reason is that a lower report would not increase the trade but it would increase the cost associated with the bias since $x_{pool}^R \leq E\left[\widetilde{x}|\psi\right]$ for any $\psi \in \left[\psi',\underline{\psi}\right]$. Since the type $\psi = \underline{\psi}$ does not want to mimic any type $\psi > \underline{\psi}$ (this follows because the type $\underline{\psi}$ does not want to mimic any $\psi > \underline{\psi}$ in the equilibrium without short sale constraints), any type $\psi < \underline{\psi}$ also does not want to mimic any type $\psi > \underline{\psi}$ (it is even more costly for $\psi < \underline{\psi}$ to mimic any type $\psi > \underline{\psi}$). Hence, an analyst that observes any $\psi \in \left[\psi',\underline{\psi}\right]$ does not want to issue a report $x^R > x_{pool}^R$.

For any $\psi < \psi'$ the equilibrium forecast is $E\left[\widetilde{x}|\psi\right]$. This forecast minimizes his expected cost from forecast errors and induces the maximum trade. Hence, the analyst has no incentive to deviate.

7.5 Unknown analyst's objective function

Suppose analysts observe ψ and v, where $\tilde{v} \sim N(0, \tau_v)$ and analyst's objective is to maximize

$$c_A |D_1(x^R) - D_0| - \int g(x^R - \widetilde{x} - \upsilon) f(x|\psi) dx$$

Before we show the existence of an equilibrium to this game, we show that there exists an equilibrium to a modified game. In the modified game, v is common knowledge, i.e. both the analyst and the investor learn v at the beginning of the period. For each realization of v, we solve for the equilibrium forecast and equilibrium trade. We then relax the assumption that the investor observes v.

In the modified game, not only the analyst's biasing cost function $g(x^R - \tilde{x} - v)$ but also his benefit is a function of v. The analyst's objective function is given by

$$c_A \left| \frac{x^R - \widehat{b}(x^R, \upsilon) + \upsilon - P_0'}{\rho' Var(x|\psi)} - D_0 \right| - \int g(x^R - \widetilde{x} - \upsilon) f(x|\psi) dx$$

where $x^R - \hat{b}\left(x^R, v\right) = E\left[\tilde{x}|\psi\right]$. Since v is common knowledge and the demand is linear in the conditional expectation, the modified game has a solution similar to the one established in Propositions 1 and 2. Hence, there exists a unique fully separating equilibrium. In such a fully separating equilibrium, the investor can infer ψ from the analyst's forecast. Since $x^R(\psi,v) = E\left[x|\psi\right] + b\left(\psi,v\right)$, solving for the optimal bias $b\left(\psi,v\right)$ is equivalent to solving for the optimal forecast $x^R\left(\psi,v\right)$. One can show that for any v, the equilibrium bias satisfies the following equation.²¹

$$b(\psi, \upsilon) = b\left(\psi + \frac{\tau_x + \tau_{\varepsilon}}{\tau_{\varepsilon}}\upsilon, 0\right) + \upsilon$$

First, note that $x^{R}(\psi, v)$ depends on (ψ, v) only through y, where $y = E[\tilde{x}|\psi] + v$. To see this consider

$$\begin{aligned} x^R \left(\psi, \upsilon \right) &= E \left[x | \psi \right] + b \left(\psi, \upsilon \right) \\ &= E \left[x | \psi \right] + b \left(\psi + \frac{\tau_x + \tau_\varepsilon}{\tau_\varepsilon} \upsilon, 0 \right) + \upsilon \\ &= y + b \left(\frac{\tau_\varepsilon}{\tau_\varepsilon + \tau_x} y, 0 \right) \\ &\equiv x^R \left(y \right) \end{aligned}$$

For notational convenience let $q(x^R) = y$ denote the inverse function of $x^R(y)$.

Since $x^R(y)$ is an equilibrium for a known v, we have $E[x|x^R,v]=q(x^R)-v$ in equilibrium. With that, we can rewrite the analyst's optimization problem as

$$c_{A} \left| \frac{E\left[x|x^{R}, \upsilon\right] + \upsilon - P_{0}'}{\rho' Var\left(x|\psi\right)} - D_{0} \right| - \int g\left(x^{R} - \widetilde{x} - \upsilon\right) f\left(x|\psi\right) dx$$

$$= c_{A} \left| \frac{q\left(x^{R}\right) - P_{0}'}{\rho' Var\left(x|\psi\right)} - D_{0} \right| - \int g\left(x^{R} - \widetilde{x} - \upsilon\right) f\left(x|\psi\right) dx$$

$$(12)$$

Next, we show that this $x^{R}(y)$ is the analyst's equilibrium forecasting strategy not only in the modified game but also in the original game when v is unknown and the coefficients have their original values. To show that $x^{R}(y)$ is the equilibrium forecast, we assume that if the investor believes that the analyst's forecasting strategy is $x^{R}(y)$ then it is in fact optimal

²¹We can think of the analyst bias as a function of $E[\widetilde{x}|\psi]$ and v which yields $b(E[\widetilde{x}|\psi],v) = b(E[\widetilde{x}|\psi]+v,0)+v$. To see that note that the cost minimizing bias increases by v while the conditional expectation of x for which the analyst's benefit is zero decreases by v.

for the analyst to follow this strategy. If the investor believes that the analyst forecasts according to $x^{R}(y)$ then the investor's trade is given by

$$\left| \frac{E\left[x|x^R\right] - P_0}{\rho Var\left(x|y\right)} - D_0 \right|$$

where the posterior expectation and variance are

$$E\left[x|x^{R}\right] = E\left[x|q\left(x^{R}\right) = y\right] = \frac{\tau_{\varepsilon}\tau_{\upsilon}q\left(x^{R}\right) + (\tau_{x} + \tau_{\varepsilon})\tau_{x}\mu_{x}}{\tau_{\varepsilon}\tau_{\upsilon} + (\tau_{x} + \tau_{\varepsilon})\tau_{x}}$$

$$Var\left(x|x^{R}\right) = Var\left(x|y\right) = \frac{1}{\tau_{x}} - \frac{\tau_{\varepsilon}\tau_{\upsilon}}{\tau_{\varepsilon}\tau_{\upsilon} + (\tau_{x} + \tau_{\varepsilon})\tau_{x}}\frac{\tau_{\varepsilon}}{\tau_{x} + \tau_{\varepsilon}}\frac{1}{\tau_{x}}$$

Hence, the analyst's optimization problem is equivalent to (12) if

$$\rho' = \rho \frac{\tau_{\varepsilon} \tau_{v} + (\tau_{x} + \tau_{\varepsilon}) \tau_{x}}{\tau_{\varepsilon} \tau_{v}} \frac{Var(x|y)}{Var(x|\psi)}$$

$$P'_{0} = P_{0} \frac{\tau_{\varepsilon} \tau_{v} + (\tau_{x} + \tau_{\varepsilon}) \tau_{x}}{\tau_{\varepsilon} \tau_{v}} - \frac{(\tau_{x} + \tau_{\varepsilon})}{\tau_{\varepsilon} \tau_{v}} \tau_{x} \mu_{x}$$

To verify that the modified game is equivalent to the original game we plug the above values for ρ' and P'_0 into the analyst's optimization problem in (12).

$$c_{A} \left| \frac{q(x^{R}) - P_{0}'}{\rho' Var(x|\psi)} - D_{0} \right|$$

$$= c_{A} \left| \frac{q(x^{R}) - \left(P_{0} \frac{\tau_{\varepsilon} \tau_{\upsilon} + (\tau_{x} + \tau_{\varepsilon}) \tau_{x}}{\tau_{\varepsilon} \tau_{\upsilon}} - \frac{(\tau_{x} + \tau_{\varepsilon})}{\tau_{\varepsilon} \tau_{\upsilon}} \tau_{x} \mu_{x}\right)}{\rho^{\frac{\tau_{\varepsilon} \tau_{\upsilon} + (\tau_{x} + \tau_{\varepsilon}) \tau_{x}}{\tau_{\varepsilon} \tau_{\upsilon}} \frac{Var(x|y)}{Var(x|\psi)} Var(x|\psi)} - D_{0} \right|$$

$$= c_{A} \left| \frac{q(x^{R}) \frac{\tau_{\varepsilon} \tau_{\upsilon} + (\tau_{x} + \tau_{\varepsilon}) \tau_{x}}{\tau_{\varepsilon} \tau_{\upsilon} + (\tau_{x} + \tau_{\varepsilon}) \tau_{x}} - P_{0} + \frac{(\tau_{x} + \tau_{\varepsilon}) \tau_{x} \mu_{x}}{\tau_{\varepsilon} \tau_{\upsilon} + (\tau_{x} + \tau_{\varepsilon}) \tau_{x}}} - D_{0} \right|$$

$$= c_{A} \left| \frac{E[x|x^{R}] - P_{0}}{\rho Var(x|y)} - D_{0} \right|$$

Since the analyst's optimization problem is equivalent to (12), the analyst's optimal forecast will also be $x^{R}(y)$. Hence, the investor's beliefs $q(x^{R})$ are consistent with the analyst's forecasting behavior. Thus, $x^{R}(y)$ and $q(x^{R})$ constitute an equilibrium.

7.6 Proof of Proposition 3

There is a unique fully separating equilibrium. For $\psi > \psi^*$. The analyst maximizes

$$c_{A} \frac{x^{R} - \widehat{b}(x^{R}) - P_{0}}{\rho Var(x|\psi)} - \frac{1}{2} E\left[\left(x^{R} - \widetilde{x}\right)^{2} \middle| \psi\right]$$

where $\hat{b}(x^R)$ denotes investor's beliefs about the bias if he observes x^R . Then, the FOC is

$$c_{A} \frac{1 - \frac{\partial \widehat{b}(x^{R})}{\partial x^{R}}}{\rho Var(\widetilde{x}|\psi)} - E\left[x^{R} - \widetilde{x}|\psi\right] = 0$$

$$c_{A} \frac{1 - \frac{\partial \widehat{b}(x^{R})}{\partial x^{R}}}{\rho Var(\widetilde{x}|\psi)} - \left(x^{R} - E\left[\widetilde{x}|\psi\right]\right) = 0$$

$$c_{A} \frac{1 - \frac{\partial \widehat{b}(x^{R})}{\partial x^{R}}}{\rho Var(\widetilde{x}|\psi)} - b(\psi) = 0$$
(13)

In equilibrium, investor's beliefs about the bias has to be correct, i.e.,

$$\widehat{b}\left(x^{R}\left(\psi\right)\right) = b\left(\psi\right).$$

We also know that (boundary condition)

$$b(\psi^*) = 0$$

With that,

$$\frac{\partial \widehat{b}\left(x^{R}\right)}{\partial x^{R}} = \frac{\partial b\left(\psi\right)}{\partial \psi} \frac{1}{\frac{\partial x^{R}}{\partial \psi}} = b'\left(\psi\right) \frac{1}{\frac{\partial E\left[\widetilde{x}\left|\psi\right|\right]}{\partial \psi} + b'\left(\psi\right)}$$

where $\frac{\partial E[\widetilde{x}|\psi]}{\partial \psi}=\frac{\tau_\varepsilon}{\tau_x+\tau_\varepsilon}$ and the FOC becomes

$$c_{A} \frac{1 - b'(\psi) \frac{1}{\frac{\partial E\left[\widetilde{x}|\psi\right]}{\partial \psi} + b'(\psi)}}{\rho Var\left(\widetilde{x}|\psi\right)} - b\left(\psi\right) = 0$$

$$c_{A} \left(\frac{\partial E\left[\widetilde{x}|\psi\right]}{\partial \psi} + b'(\psi)\right) - c_{A}b'(\psi) = b\left(\psi\right) \rho Var\left(\widetilde{x}|\psi\right) \left(\frac{\partial E\left[\widetilde{x}|\psi\right]}{\partial \psi} + b'(\psi)\right)$$

$$c_{A} \frac{\partial E\left[\widetilde{x}|\psi\right]}{\partial \psi} = b\left(\psi\right) \rho Var\left(\widetilde{x}|\psi\right) \left(\frac{\partial E\left[\widetilde{x}|\psi\right]}{\partial \psi} + b'(\psi)\right)$$

$$0 = b\left(\psi\right) + b'\left(\psi\right) b\left(\psi\right) \frac{\tau_{x} + \tau_{\varepsilon}}{\tau_{z}} - \frac{c_{A}\left(\tau_{x} + \tau_{\varepsilon}\right)}{\rho}$$

(Suppose there are two equilibria. Since we have the boundary condition, it must be the case that there is an interval of types for which $b(\psi)$ increases at a higher rate in one of the equilibria. In order for that to be an equilibrium it must be that the conditional expectation is increasing in the report at a higher rate as well (marginal cost = marginal benefit). However, if the bias increases at a higher rate, this implies that the conditional expectation must be increasing in the report at a lower rate. Hence, there cannot be two equilibria.)

Solution to differential equation. We need to solve

$$b(\psi) + k_1 b(\psi) * b'(\psi) + k_2 = 0$$

where $k_1 = \frac{\tau_x + \tau_{\varepsilon}}{\tau_{\varepsilon}}$ and $k_2 = -\frac{c_A(\tau_x + \tau_{\varepsilon})}{\rho}$. The boundary condition is $b(\psi^*) = 0$ where $\psi^* = P_0$. It follows from the Fundamental Theorem of Differential Equations that there is a unique solution. The unique solution to this differential equation is (from Mathematica)

$$b\left(\psi\right) = -k_2\left(1 + PL\left(-e^{\frac{\psi - P_0}{k_1 k_2} - 1}\right)\right)$$

and hence

$$b(\psi) = \frac{c_A(\tau_x + \tau_\varepsilon)}{\rho} \left(1 + PL\left(-e^{-\frac{(\psi - P_0)\tau_\varepsilon\rho}{c_A(\tau_x + \tau_\varepsilon)^2} - 1} \right) \right)$$
(14)

where $PL\left(\cdot\right)$ denotes the ProductLog function. The ProductLog function does not have a closed form solution but is defined as the solution to the following differential equation: $f'\left(x\right) = \frac{f(x)}{x(1+f(x))}.$

Alternatively from (13), we can solve the following differential equation.

$$1 - b'\left(x^R\right) - b\left(x^R\right)\frac{\rho}{c_A}Var\left(\widetilde{x}|\psi\right) = 0$$
(15)

The boundary condition to this differential solution is

$$b\left(x^{R}\left(\psi\right)\right)\big|_{\psi=\psi^{*}} = b\left(E\left[\widetilde{x}|\psi^{*}\right] + b\left(\psi^{*}\right)\right) = b\left(E\left[\widetilde{x}|\psi^{*}\right]\right) = b\left(x^{*}\right) = b\left(\frac{\mu_{x}\tau_{x} + P_{0}\tau_{\varepsilon}}{\tau_{x} + \tau_{\varepsilon}}\right) = 0$$

The unique solution to this differential equation is

$$b(x^R) = \frac{1}{a_0} \left(1 - e^{a_0(a_1 - x^R)} \right)$$

where $a_0 = \frac{\rho}{c_A} Var\left(\widetilde{x}|\psi\right) = \frac{\rho}{c_A(\tau_x + \tau_{\varepsilon})}$ and $b\left(a_1\right) = 0$. Hence,

$$b\left(x^{R}\right) = \frac{c_{A}\left(\tau_{x} + \tau_{\varepsilon}\right)}{\rho} \left(1 - e^{\frac{\rho}{c_{A}\left(\tau_{x} + \tau_{\varepsilon}\right)}\left(\frac{\mu_{x}\tau_{x} + P_{0}\tau_{\varepsilon}}{\tau_{x} + \tau_{\varepsilon}} - x^{R}\right)}\right)$$

Next, we check that the solution for $b(\psi)$ and $b(x^R)$ are consistent in the sense that $b(x^R(\psi)) = b(\psi)$.

$$b\left(x^{R}\left(\psi\right)\right) = \frac{c_{A}\left(\tau_{x} + \tau_{\varepsilon}\right)}{\rho} \left(1 - e^{\frac{\rho}{c_{A}(\tau_{x} + \tau_{\varepsilon})}\left(\frac{\mu_{x}\tau_{x} + P_{0}\tau_{\varepsilon}}{\tau_{x} + \tau_{\varepsilon}} - x^{R}(\psi)\right)}\right)$$

$$= \frac{c_{A}\left(\tau_{x} + \tau_{\varepsilon}\right)}{\rho} \left(1 - e^{\frac{\rho}{c_{A}(\tau_{x} + \tau_{\varepsilon})}\left(\frac{\mu_{x}\tau_{x} + P_{0}\tau_{\varepsilon}}{\tau_{x} + \tau_{\varepsilon}} - E[\tilde{x}|\psi] - b(\psi)\right)}\right)$$

$$= \frac{c_{A}\left(\tau_{x} + \tau_{\varepsilon}\right)}{\rho} \left(1 - e^{\frac{\rho}{c_{A}(\tau_{x} + \tau_{\varepsilon})}\left(\frac{\mu_{x}\tau_{x} + P_{0}\tau_{\varepsilon}}{\tau_{x} + \tau_{\varepsilon}} - \frac{\tau_{\varepsilon}\psi + \tau_{x}\mu_{x}}{\tau_{x} + \tau_{\varepsilon}} - b(\psi)\right)}\right)$$

$$= \frac{c_{A}\left(\tau_{x} + \tau_{\varepsilon}\right)}{\rho} \left(1 - e^{\frac{\rho}{c_{A}(\tau_{x} + \tau_{\varepsilon})}\left(\frac{\tau_{\varepsilon}}{\tau_{x} + \tau_{\varepsilon}}(P_{0} - \psi) - b(\psi)\right)}\right)$$

Solution to that implicit equation from Mathematica

$$b(\psi) = \frac{1 + PL\left(-e^{\frac{\rho}{c_A(\tau_x + \tau_{\varepsilon})}\left(\frac{\tau_{\varepsilon}}{\tau_x + \tau_{\varepsilon}}(P_0 - \psi)\right) - 1}\right)}{\frac{\rho}{c(\tau_x + \tau_{\varepsilon})}}$$

$$= \frac{c_A\left(\tau_x + \tau_{\varepsilon}\right)}{\rho}\left(1 + PL\left(-e^{\frac{\rho}{c_A(\tau_x + \tau_{\varepsilon})}\left(-\frac{\tau_{\varepsilon}}{\tau_x + \tau_{\varepsilon}}(\psi - P_0)\right) - 1}\right)\right)$$
(16)

So, we have that (14) and (16) are identical.

Finally, we derive the solution for $\psi < \psi^*$ and $x^R < x^*$. The differential equation for $b(\psi)$ is

$$-c_{A} \frac{1 - b'(\psi) \frac{1}{\frac{\partial E\left[\widetilde{x}|\psi\right]}{\partial \psi} + b'(\psi)}}{\rho Var\left(\widetilde{x}|\psi\right)} - b(\psi) = 0$$

$$-c_{A} \frac{\partial E\left[\widetilde{x}|\psi\right]}{\partial \psi} = b(\psi) \rho Var\left(\widetilde{x}|\psi\right) \left(\frac{\partial E\left[\widetilde{x}|\psi\right]}{\partial \psi} + b'(\psi)\right)$$

$$0 = b(\psi) + b'(\psi) b(\psi) \frac{\tau_{x} + \tau_{\varepsilon}}{\tau_{\varepsilon}} + \frac{c_{A}(\tau_{x} + \tau_{\varepsilon})}{\rho}$$

The boundary condition remains unchanged and hence

$$b\left(\psi\right) = -\frac{c_A\left(\tau_x + \tau_\varepsilon\right)}{\rho} \left(1 + PL\left(-e^{-\frac{\left(P_0 - \psi\right)\tau_\varepsilon\rho}{c_A\left(\tau_x + \tau_\varepsilon\right)^2} - 1}\right)\right) \text{ for } \psi < \psi^*$$

Similarly, the differential equation for $b(x^R)$ is

$$1 - b'(x^R) + b(x^R) \frac{\rho}{c_A} Var(\widetilde{x}|\psi) = 0$$

The unique solution to this differential equation is

$$b\left(x^{R}\right) = -\frac{c_{A}\left(\tau_{x} + \tau_{\varepsilon}\right)}{\rho} \left(1 - e^{\frac{\rho}{c_{A}(\tau_{x} + \tau_{\varepsilon})}\left(x^{R} - \frac{\mu_{x}\tau_{x} + P_{0}\tau_{\varepsilon}}{\tau_{x} + \tau_{\varepsilon}}\right)}\right) \text{ for } \psi < \psi^{*}$$

SOC to the analyst's forecasting problem.

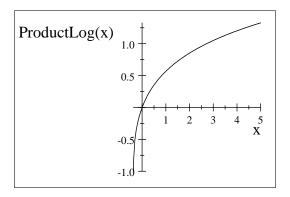
$$\frac{\partial}{\partial x^{R}} \left(c_{A} \frac{1 - \frac{\partial b(x^{R})}{\partial x^{R}}}{\rho Var(x|\psi)} - \left(x^{R} - E[x|\psi] \right) \right) = -c_{A} \frac{\frac{\partial^{2} b(x^{R})}{\partial (x^{R})^{2}}}{\rho Var(x|\psi)} - 1$$

$$= -c_{A} \frac{-\frac{\rho}{c(\tau_{x} + \tau_{\varepsilon})} e^{\frac{\rho}{c_{A}(\tau_{x} + \tau_{\varepsilon})} \left(\frac{\mu_{x} \tau_{x} + P_{0} \tau_{\varepsilon}}{\tau_{x} + \tau_{\varepsilon}} - x^{R} \right)}}{\rho Var(x|\psi)} - 1$$

$$= e^{\frac{\rho}{c_{A}(\tau_{x} + \tau_{\varepsilon})} \left(\frac{\mu_{x} \tau_{x} + P_{0} \tau_{\varepsilon}}{\tau_{x} + \tau_{\varepsilon}} - x^{R} \right)} - 1 < 0$$

7.6.1 ProductLog(x)

The ProductLog (x) function (also known as Lambert-W or Omega function) is the solution to the differential equation $f'(x) = \frac{f(x)}{x(1+f(x))}$. It is defined to any $x \geq -e^{-1}$, where $\Pr{oductLog(-e^{-1})} = -1$, $\Pr{oductLog(0)} = 0$ and $\lim_{x\to\infty} \Pr{oductLog(x)} = \infty$. The chart below demonstrates the $\Pr{oductLog(x)}$ function.



7.7 Proofs of Empirical Predictions

For simplicity we present the proof for the empirical predictions under the assumption that $c_I = 0$. Similar arguments can be made for the more general case of $c_I \ge 0$.

7.7.1 Proof of Corollary 1 - Expected bias is positive

Suppose that the cost function $g(x^R - x)$ is symmetric around zero.

Then $b(\psi) = -b(2\psi^* - \psi)$. Given that $\psi^* < \mu_x$ and the symmetry of the distribution function around μ_x the pdf of ψ is higher than the pdf of $2\psi^* - \psi$ for any $\psi > \psi^*$. Hence the expected bias is positive.

7.7.2 Proof of Corollary 2 – Comparative statics with respect to c_A

Part (a). The analyst's FOC as given in (8) is

$$\begin{cases} c_{A} \frac{\frac{\tau_{\varepsilon}}{\tau_{\varepsilon} + \tau_{x}}}{\rho Var(\widetilde{x}|\psi) \left(\frac{\tau_{\varepsilon}}{\tau_{\varepsilon} + \tau_{x}} + b'(\psi)\right)} - \frac{\partial}{\partial b} E\left[g\left(E\left[\widetilde{x}|\psi\right] + b - \widetilde{x}\right)|\psi\right] = 0 & \text{if } \psi > \psi^{*} \\ -c_{A} \frac{\tau_{\varepsilon}}{\rho Var(\widetilde{x}|\psi) \left(\frac{\tau_{\varepsilon}}{\tau_{\varepsilon} + \tau_{x}} + b'(\psi)\right)} - \frac{\partial}{\partial b} E\left[g\left(E\left[\widetilde{x}|\psi\right] + b - \widetilde{x}\right)|\psi\right] = 0 & \text{if } \psi^{*} > \psi \end{cases}$$

Applying the implicit function theorem to the LHS of the analyst's FOC yields

$$\frac{\partial b\left(\psi\right)}{\partial c_{A}} = -\frac{\frac{\partial FOC}{\partial c_{A}}}{\frac{\partial FOC}{\partial b}} = -\frac{\frac{\frac{\tau_{\varepsilon}}{\tau_{\varepsilon} + \tau_{x}}}{\rho Var(\widetilde{x}|\psi)\left(\frac{\tau_{\varepsilon}}{\tau_{\varepsilon} + \tau_{x}} + b'(\psi)\right)}}{\frac{\partial FOC}{\partial b}} > 0 \text{ if } \psi > \psi^{*}$$

$$\frac{\partial b\left(\psi\right)}{\partial c_{A}} = -\frac{\frac{\partial FOC}{\partial c_{A}}}{\frac{\partial FOC}{\partial b}} = -\frac{\frac{\tau_{\varepsilon}}{\tau_{\varepsilon} + \tau_{x}}}{\frac{\tau_{\varepsilon} + \tau_{x}}{\rho Var(\widetilde{x}|\psi)\left(\frac{\tau_{\varepsilon}}{\tau_{\varepsilon} + \tau_{x}} + b'(\psi)\right)}}{\frac{\partial FOC}{\partial b}} < 0 \text{ if } \psi > \psi^{*}$$

where $\frac{\partial FOC}{\partial b}$ < 0 because we are at a local maximum.

Part (b). We first want to show that $\frac{\partial E[-FE]}{\partial c_A} = \frac{\partial E[b(\psi)]}{\partial c_A} > 0$. For any symmetric cost function we have,

$$E\left[b\left(\psi\right)\right] = \int_{\psi^*}^{\infty} b\left(\psi; c_A\right) \left[f\left(\psi\right) - f\left(2\psi^* - \psi\right)\right] d\psi$$

and hence

$$\frac{\partial E\left[b\left(\psi\right)\right]}{\partial c_{A}} = \int_{\psi^{*}}^{\infty} \frac{\partial b\left(\psi; c_{A}\right)}{\partial c_{A}} \left[f\left(\psi\right) - f\left(2\psi^{*} - \psi\right)\right] d\psi > 0$$

because $\frac{\partial b(\psi;c_A)}{\partial c_A} > 0$ and $[f(\psi) - f(2\psi^* - \psi)]$ as well as ψ^* are independent of c_A .

Next we show that $\frac{\partial E[FE^2]}{\partial c_A} > 0$. Since the magnitude of $b(\psi)$ is increasing in c_A for all $\psi \neq \psi^*$, the expected squared bias is also increasing in c_A . The conditional variance of x does not depend on c_A .

Part (c). Noting that $\frac{\frac{\tau_{\varepsilon}}{\tau_{\varepsilon}+\tau_{x}}}{\rho Var(\tilde{x}|\psi)\left(\frac{\tau_{\varepsilon}}{\tau_{\varepsilon}+\tau_{x}}+b'(\psi)\right)}$ is finite, the analyst's marginal trading commissions approaches 0 as $c_{A} \to 0$. From this it follows that in equilibrium his expected marginal cost from biasing his forecast also approaches zero as $c_{A} \to 0$, and hence $\lim_{c_{A} \to 0} b(\psi) = 0$.

7.7.3 Comparative statics of the bias with respect τ_{ε}

Suppose that $\frac{\partial g(FE)}{\partial FE} = \frac{\partial g(b+\varepsilon)}{\partial b}$ is weakly convex in b (note that for a given b, $\frac{\partial g(b+\varepsilon)}{\partial b}$ is independent of ψ).

The marginal expected cost is decreasing in τ_{ε} . An decrease in τ_{ε} may be achieved by a series of Mean Preserving Spreads (MPS). Given the weak convexity of $\frac{\partial g(b+\varepsilon)}{\partial b}$, the expected value of $\frac{\partial g(b+\varepsilon)}{\partial b}$ is weakly increasing following any MPS and hence following a decrease in τ_{ε} . This implies that $\frac{\partial^2 E[g(b+\varepsilon)]}{\partial \tau_{\varepsilon} \partial b} \leq 0$.

In order to show that for any two precisions of private signals, $\tau_{\varepsilon}^1 > \tau_{\varepsilon}^0$ and any $\psi > \psi^*$ the bias of type 1 is higher than the boas of type 0, we first show that for ψ going to infinity

the bias of type 1 is higher than that of type 0, and next we show that the biases of the two types cannot intersect.

For ψ going to infinity the bias is bounded and converges to a constant, so the investor's expectation about \widetilde{x} is linear in the forecast with a slope of one. As a result of this and $\tau_{\varepsilon}^{1} > \tau_{\varepsilon}^{0}$ the marginal benefit (which equals, in equilibrium, the marginal expected cost) of type 1 from biasing his forecast is higher. The fact that $\frac{\partial^{2}E[g(b+\varepsilon)]}{\partial \tau_{\varepsilon}\partial b} \leq 0$ implies that $\frac{\partial E[g(b+\varepsilon)]}{\partial b}$ is higher for type 1 only if the bias of type 1 when ψ goes to infinity is higher.

Suppose that the bias functions of the two types intersect. Since the bias is zero at ψ^* for both types, then there must be a $\psi \geq \psi^*$ for which the two biases are identical and the bias function of type 0 has a higher slope. At this point where the biases are identical, the marginal expected cost is higher for type 0 since $\frac{\partial^2 E[g(b+\varepsilon)]}{\partial \tau_{\varepsilon} \partial b} \leq 0$. However, the marginal benefit from biasing the forecast is lower for type 0 because both the slope of his bias function is higher and the precision of his forecast is lower. This contradiction implies that the bias of type 1 is always higher than the bias of type 0 for any $\psi > \psi^*$. A similar argument can be made for all $\psi < \psi^*$.

7.7.4 Proof of Corollary 5 – Expected squared forecast error and Expected forecast error

We start by proving part (b).

Part (b). First we show that $\lim_{\tau_{\varepsilon}\to\infty} E\left[FE\right] = -\infty$. For $\tau_{\varepsilon}\to\infty$ the distribution of ψ converges to the distribution of x, i.e., for any $\psi>\psi^*\lim_{\tau_{\varepsilon}\to\infty}\left[f_{\psi}\left(\psi\right)-f_{\psi}\left(2\psi^*-\psi\right)\right]=\left[f_{x}\left(\psi\right)-f_{x}\left(2\psi^*-\psi\right)\right]>0$ where $f_{i}\left(\cdot\right)$ denotes the pdf of the random variable i. Next, we show that $\lim_{\tau_{\varepsilon}\to\infty}b\left(\psi\right)=\infty$ for all $\psi>\psi^*$.

As $\tau_{\varepsilon} \to \infty$ the analyst's marginal benefit from biasing his forecast (which is the marginal trading commissions generated by his forecast) goes to infinity as well, as long as the bias function has a finite slope (which holds for all $\psi \neq \psi^*$). The reason is that following the analyst's forecast, the investor is not exposed to any risk (in our fully revealing equilibrium) and hence his trade is infinitely sensitive to the analyst's forecast. In equilibrium, the analyst's marginal benefit equals his marginal expected cost, and hence his marginal expected cost for $\tau_{\varepsilon} \to \infty$ must be infinite as well. Since this holds for any ψ , we have $\lim_{\tau_{\varepsilon} \to \infty} b(\psi) = \infty$ for all $\psi > \psi^*$.

Recall that $\lim_{\tau_{\varepsilon}\to\infty} \left[f_{\psi}\left(\psi\right) - f_{\psi}\left(2\psi^* - \psi\right)\right] > 0$ and due to the symmetry of $g\left(\cdot\right)$ we have $E\left[b\left(\psi\right)\right] = \int_{\psi^*}^{\infty} b\left(\psi\right) \left[f\left(\psi\right) - f\left(2\psi^* - \psi\right)\right] d\psi$. From this it follows that $\lim_{\tau_{\varepsilon}\to\infty} E\left[b\left(\psi\right)\right] = \infty$.

Part (a). Next, we show that $\lim_{\tau_{\varepsilon}\to\infty} E\left[FE^2\right] = \infty$. Consider that $E\left[FE^2\right] = E\left[b\left(\psi\right)^2\right] + \frac{1}{\tau_x + \tau_{\varepsilon}}$. As $\tau_{\varepsilon} \to \infty$, for any $\psi \neq \psi^* \ b\left(\psi\right)^2$ goes to infinity and the distribution of ψ converges to the distribution of x (i.e. there is no mass point at $\psi = \psi^*$). Hence, while integrating over ψ the only point in which the bias is bounded (=0) is $\psi = \psi^*$ which has zero mass.

Part (c). Recall that the bias is bounded for any set of parameters. In part (a) of corollary 2 we showed that for any ψ $\frac{\partial |b(\psi;c_A)|}{\partial c_A} > 0$. Part (c) of corollary 2 shows that $\lim_{c_A \to 0} b(\psi; c_A) = 0$. Hence the upper (lower) bound of the bias is increasing (decreasing) in c_A and converges to zero as c_A goes to zero.

Note that: $E\left[FE^2\right] = E\left[b\left(\psi;c_A\right)^2\right] + \frac{1}{\tau_x + \tau_\varepsilon}$. We want to show that there exists c_A^* such that for any $c_A < c_A^*$, an increase of τ_ε by δ is followed by a decrease in the variance which is greater than the increase in the expected squared bias. Assume without loss of generality that $\lim_{\psi \to \infty} b\left(\psi;c_A\right)|_{\tau_\varepsilon + \delta} \ge -\lim_{\psi \to -\infty} b\left(\psi;c_A\right)|_{\tau_\varepsilon + \delta}$.

$$0 < E \left[b \left(\psi; c_A \right)^2 \right] \Big|_{\tau_{\varepsilon}}$$

$$E \left[b \left(\psi; c_A \right)^2 \right] \Big|_{\tau_{\varepsilon} + \delta} < \left(\lim_{\psi \to \infty} b \left(\psi; c_A \right) \Big|_{\tau_{\varepsilon} + \delta} \right)^2$$

From that we know that

$$E\left[b\left(\psi;c_{A}\right)^{2}\right]\Big|_{\tau_{\varepsilon}+\delta}-E\left[b\left(\psi;c_{A}\right)^{2}\right]\Big|_{\tau_{\varepsilon}}<\left(\lim_{\psi\to\infty}b\left(\psi;c_{A}\right)\Big|_{\tau_{\varepsilon}+\delta}\right)^{2}$$

Note that the upper bound is monotone increasing in c_A and continuous. For $c_A \to 0$ the upper bound goes to zero as well.

When τ_{ε} increases by δ , the conditional variance of x given ψ decreases by

$$\frac{1}{\tau_x + \tau_\varepsilon} - \frac{1}{\tau_x + \tau_\varepsilon + \delta}$$

Hence, there exist a c_A^* such that

$$\frac{1}{\tau_{x} + \tau_{\varepsilon}} - \frac{1}{\tau_{x} + \tau_{\varepsilon} + \delta} = \left(\lim_{\psi \to \infty} b\left(\psi; c_{A}^{*}\right)\right|_{\tau_{\varepsilon} + \delta}\right)^{2}$$

and hence for any $c_A < c_A^*$ we have

$$\left(\lim_{\psi \to \infty} b\left(\psi; c_A\right)|_{\tau_{\varepsilon} + \delta}\right)^2 - \left(\frac{1}{\tau_x + \tau_{\varepsilon}} - \frac{1}{\tau_x + \tau_{\varepsilon} + \delta}\right) < 0$$

that is, the increase in the expected squared bias is smaller than the decrease in the conditional variance. As a result the expected squared forecast error decreases as τ_{ε} increases by δ .

7.8 Optimal Cost Function Coefficient

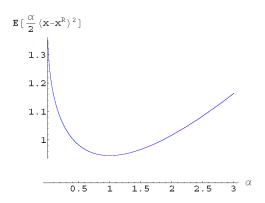
We show for the case of a quadratic cost function, that the analyst's preferred cost function coefficient is not arbitrarily small. To capture this idea, we allow the coefficient of the analyst's cost function to be different from 1. In particular, we assume that the analyst's cost from forecast errors are $\frac{\alpha}{2} (x^R - \tilde{x})^2$ for $\alpha \geq 0$. That is the analyst's payoff function is given by

$$u^{A}\left(x^{R},\psi\right)=c_{A}\left|D_{1}\left(x^{R}\right)-D_{0}\right|-\frac{\alpha}{2}E\left[\left(x^{R}-\widetilde{x}\right)^{2}\right|\psi\right].$$

In the following, we show that the analyst would neither choose $\alpha = 0$ nor α arbitrarily close to zero if he were able to choose α prior to observing his private signal ψ . The result that the analyst prefers a parameter α that is bounded away from zero suggests that ex-ante the analyst would like to commit to non-zero costs from forecast errors. This provides further justification for the model's assumption that the analyst incurs costs from forecast errors.

Claim 8
$$\underset{\alpha}{\operatorname{arg\,max}} E\left[u^{A}\left(x^{R}\left(\widetilde{\psi}\right),\widetilde{\psi};\alpha\right)\right] > 0.$$

The proof of the Claim is based on the following argument. First, note that if $\alpha=0$, biasing his forecast is costless to the analyst resulting in a bubbling equilibrium in which the analyst does not generate any trade. Therefore, the analyst's expected utility for $\alpha=0$ is zero. For any $\alpha>0$, there exists a fully separating equilibrium in which the investor can perfectly infer the analyst's private information. As a result, the investor's trading behavior and the analyst's expected trading commissions do not depend on α as long as $\alpha>0$. Therefore, which α is preferred by the analyst depends only on the expected cost from forecast errors. One might expect that the analyst would always prefer a lower value of α . However, this is not the case. The reason is that the magnitude of the analyst's bias converges to ∞ as α converges to zero in a way that the analyst's expected cost from forecast errors are strictly positive, i.e. $\lim_{\alpha\downarrow0} E\left[\frac{\alpha}{2}\left(x^R\left(\widetilde{\psi}\right)-\widetilde{x}\right)^2\right]>0$. Next, we show that $\lim_{\alpha\downarrow0} \frac{\partial E\left[\frac{\alpha}{2}\left(x^R(\widetilde{\psi})-\widetilde{x}\right)^2\right]}{\partial\alpha}=\infty$ and $\lim_{\alpha\uparrow\infty} E\left[\frac{\alpha}{2}\left(x^R\left(\widetilde{\psi}\right)-\widetilde{x}\right)^2\right]=\infty$. These results



together suggest that the analyst's preferred value of α is strictly greater than zero and finite. The following plot presents the analyst's expected cost from forecast error as a function of α . The parameters used to generate the plot are: $c_A = \rho = \tau_x = \tau_\varepsilon = S = 1$; $c_I = 0$ and $\mu_x = 2$.

Formal proofs are available upon request.

References

- [1] Chen Q. and W. Jiang, 2006, Analysts' Weighting of Private and Public Information, Review of Financial Studies, 19, 319-355.
- [2] Dugar A. and S. Nathan, 1995, The Effect of Investment Banking Relationships on Financial Analysts' Earnings Forecasts and Investment Recommendations, Contemporary Accounting Research, Fall 1995, 131-160.
- [3] Dye, R. A. and S. Sridhar, 2004, Reliability-Relevance Trade-Offs and the Efficiency of Aggregation, Journal of Accounting Research, 42, 51-88.
- [4] Easterwood J. and Nutt S., 1999, Inefficiency in Analysts' Earnings Forecasts: Systematic Misreaction or Systematic Optimism?, Journal of Finance, 54 (5), 1777-1797.
- [5] Friesen, G. and P. Weller, 2006, Quantifying cognitive biases in analyst earnings forecasts, Journal of Financial Markets, 9 (4), 333-365.
- [6] Grossman S.J. and J.E. Stiglitz, 1980, On the impossibility of informationally efficient markets. American Economic Review, 70, 393–408.
- [7] Gu, Z. and J. Xue, 2007, The Disciplining Role and Superiority of Independent Analysts, working paper.
- [8] Guttman, I., 2007, The Timing of Analysts' Earnings Forecasts, working paper Stanford University.
- [9] Hayes R., 1998, The Impact of Trading Commission Incentives on Analyst's Stock Coverage Decision and Earnings Forecasts, Journal of Accounting Research, 36, 299-320.
- [10] Hong H. and J. Kubik, 2003, Analyzing the Analysts: Career Concerns and Biased Earnings Forecasts, Journal of Finance, February 2003.
- [11] Irvine P., 2004, Analysts' Forecasts and Brokerage-Firm Trading, Accounting Review, 79, 125-149.
- [12] Jackson A., 2005, Trade Generation, Reputation, and Sell-Side Analysts, Journal of Finance, 60, 673-717.

- [13] Kothari, S., P., 2001, Capitals research in accounting, Journal of Accounting and Economics 31, 105-231.
- [14] Kyle, A. S., 1985, Continuous Auctions and Insider Trading, Econometrica, 53, 1315-1335.
- [15] Lim, T., 2001, Rationality and Analysts Forecast Bias, Journal of Finance, 56, 369-385.
- [16] Lin H. and M. McNichols, 1998, Underwriting Relationships, Analysts' Earnings Forecasts and Investment Recommendations, Journal of Accounting and Economics, 25, 101-127.
- [17] Michaely R. and K. Womack, 1999, Conflict of Interest and Credibility of Underwriter Analyst Recommendations, Review of Financial Studies, 12, 653-686.
- [18] Mikhail M., B. Walther and R. Willis, 1999, Does Forecast Accuracy Matter to Security Analysts?, The Accounting Review, 74, 185-200.
- [19] Morgan J. and P. Stocken, 2003, An Analysis of Stock Recommendations, RAND Journal of Economics, 34, 183-203.
- [20] Stickel, S.E, 1992, Reputation and Performance among Security Analysts, Journal of Finance, 47(5), 1811-36.